A Robust Variable Step-Size NLMS Algorithm Through A Combination of Robust Cost Functions

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Abstract—This letter introduces a new gradient-based adaptive filtering algorithm based on a cost function that is constructed by combining two robust cost functions, which are a new *tanh*-type cost function and Vega's cost function. Through the approach to combine robust cost functions, the robustness of the proposed algorithm outperforms that of other adaptive algorithms. Since the proposed algorithm is derived by combining two robust cost functions, it leads to an excellent transient and steady-state behavior in high probability of impulsive measurement noise. The proposed algorithm is tested in different probability of impulsive measurement noise.

Index Terms—Adaptive filter, robust filtering, impulsive noise, normalized least-mean-square (NLMS) algorithm.

I. INTRODUCTION

Adaptive algorithms have played major roles in various signal processing applications, such as system identification, echo cancellation, channel equalization, and control in communication networks [1]–[4]. For this, the normalized least-mean-squares (NLMS) algorithm has been widely used in many applications due to its simplicity and robustness [1], [2]. However, in most real environments, adaptive filtering applications can be influenced by various outliers including impulsive measurement noise and they cause the performance degradation of many adaptive filters. To overcome this drawback, many various schemes have been proposed [5]–[14].

When using gradient-based adaptive filtering, the next weight estimate is determined to reduce the cost function associated with the *a posteriori* output errors by using the gradient of the cost function with respect to the current weight vector, where the step size in $(0; 1]$ is usually designed to enhance the rate of convergence. Therefore, the cost function and the step size play an extremely important role in gradient-based adaptive filtering. The most common idea to handle impulsive measurement noise has been considered to design the appropriate cost function, which directly determines the performance of the algorithms.

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Various robust cost functions using the $L1$ norm have been developed for impulsive-noise environments [5]–[11]. A normalized least mean absolute deviation (NLMAD) algorithm [5], a dual sign algorithm [6], variable step-size sign algorithms [7], [8], and an affine projection sign algorithm [9] are based on the $L1$ -norm minimization, thus the algorithms use the sign of the output error to update next weight estimate. A robust mixed-norm adaptive algorithm is based on a robust cost function using the convex combination of the $L1$ and $L2$ norms [10]. A robust variable-step-size (VSS) NLMS algorithm uses a robust cost function which switches between the $L1$ and $L2$ norms [11]. It is associated with how the algorithm constrains the energy of the filter update at each iteration. Various adaptive algorithms use other robust cost functions for robustness against impulsive measurement noise [12]–[14]. When the magnitude of the output error is larger than a threshold, the Huber mixed-norm M-estimate cost function uses the $L1$ norm minimization [12], on the other hand, the Hampel three-part redescending M-estimate cost function sets the error signal as a constant value [13], [14].

In this paper, we propose a new gradient-based adaptive filtering algorithm based on a cost function that is constructed by combining two robust cost functions. One is a new cost function which is developed by investigating and modifying the *tanh* cost function and the latter is a robust cost function which Vega proposed in [11]. Through our approach to combine cost functions, the robustness of the proposed algorithm outperforms that of other adaptive algorithms. Since the proposed algorithm is derived by combining two robust cost functions, it leads to an excellent transient and steady-state behavior in high probability of impulsive measurement noise. Using simulations, the proposed algorithm is checked the robustness against impulsive measurement noise and we compared it with various robust adaptive algorithms in impulsive-noise environment.

II. NEW ROBUST COST FUNCTIONS

A. Tanh Cost Function

In system identification, the desired signal d_i is represented as

$$d_i = \mathbf{u}_i^T \mathbf{w} + \bar{v}_i \quad (1)$$

where, $\mathbf{w} \in \mathcal{R}^n$ is a coefficient vector of the unknown system, the input vector $\mathbf{u}_i$ denotes $[x_i, \dots, x_{i-n+1}]^T$, and the scalar variable $\bar{v}_i = v_i + \eta_i$, where input sequence $\{x_i\}$ is colored

with variance σ_x^2 and v_i denotes the white Gaussian measurement noise with $N(0, \sigma_v^2)$, and η_i is the impulsive measurement noise. The subscript $()^T$ denotes the vector transpose operation. Let $\hat{\mathbf{w}}_i$ be an estimate $\mathbf{w}$ at the i -th iteration and e_i be an a priori measurement error, defined as

$$e_i = d_i - \mathbf{u}_i^T \hat{\mathbf{w}}_i. \quad (2)$$

In [15], Hampel introduced the concept of the *tanh* cost function for M-estimator in robust estimation. Using the concept, Wang proposed a robust cost function using the square value of the output error for robust estimation of Principal Components Analysis (PCA) [16]. Similar to this function, we shall suggest a new cost function using the square value of the normalized instant output error with respect to the input vector as follows:

$$J_1(\hat{\mathbf{w}}_i) = \tanh\left(\left(e_i / \|\mathbf{u}_i\|\right)^2\right) = \frac{1 - \exp\left(-2\left(e_i / \|\mathbf{u}_i\|\right)^2\right)}{1 + \exp\left(-2\left(e_i / \|\mathbf{u}_i\|\right)^2\right)}, \quad (3)$$

where $\beta > 0$ is the slope. $J_1(\hat{\mathbf{w}}_i)$ is even, non-negative, and differentiable all $e_i / \|\mathbf{u}_i\|$; thus it can be a cost function. When $e_i / \|\mathbf{u}_i\|$ is small, Taylor expansions provided the approximation as follows:

$$J_1(\hat{\mathbf{w}}_i) = \left(e_i / \|\mathbf{u}_i\|\right)^2, \quad (4)$$

From which this function works like as $L2$ square-mean and the associated algorithm performs like an NLMS algorithm.

If impulsive measurement noise appears, then $e_i / \|\mathbf{u}_i\|$ becomes very large, which sends the cost function into 1, and furthermore its derivative into zero, which plays a crucial role in achieving robustness over impulsive measurement noise.

B. Proposed Robust NLMS Algorithm

Using Huber's cost function [17], Vega proposed a robust cost function as follows [11]:

$$\rho(e_i / \|\mathbf{u}_i\|) = \begin{cases} \frac{1}{2} \left(\frac{e_i}{\|\mathbf{u}_i\|} \right)^2, & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} \leq \sqrt{\delta} \\ \sqrt{\delta} \frac{|e_i|}{\|\mathbf{u}_i\|} - \frac{\delta}{2}, & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} > \sqrt{\delta}. \end{cases} \quad (5)$$

This cost function is related to the framework of the constraint of the energy of the filter update at each iteration as

$$\|\hat{\mathbf{w}}_{i+1} - \hat{\mathbf{w}}_i\|^2 \leq \delta_i, \quad (6)$$

where $\{\delta_i\}$ is a positive sequence. Using the constraint, Vega proposed a robust variable step-size NLMS (RVSS NLMS) algorithm as follows [11]:

$$\hat{\mathbf{w}}_{i+1} = \hat{\mathbf{w}}_i + \min\left[\frac{|e_i|}{\|\mathbf{u}_i\|}, \sqrt{\delta_i}\right] \text{sgn}(e_i) \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}, \quad (7)$$

$$\delta_{i+1} = \alpha \delta_i + (1 - \alpha) \min\left[\frac{e_i^2}{\|\mathbf{u}_i\|^2}, \delta_i\right].$$

Fig. 1. The proposed cost function $J(\hat{\mathbf{w}}_i)$.

To develop a robust gradient-based adaptive algorithm, which performs better than RVSS NLMS in impulsive-noise Environment, we propose a robust cost function through a combination of the cost functions $J_1(\hat{\mathbf{w}}_i)$ and $\rho(e_i / \|\mathbf{u}_i\|)$ as follows:

$$J(\hat{\mathbf{w}}_i) = \tanh\left(\rho(e_i / \|\mathbf{u}_i\|)\right) = \frac{1 - \exp\left(-2\rho(e_i / \|\mathbf{u}_i\|)\right)}{1 + \exp\left(-2\rho(e_i / \|\mathbf{u}_i\|)\right)}. \quad (8)$$

This function also satisfies the conditions of the cost function and like $J_1(\hat{\mathbf{w}}_i)$. Let us develop the adaptive algorithm base on the proposed cost function $J(\hat{\mathbf{w}}_i)$:

$$\hat{\mathbf{w}}_{i+1} = \hat{\mathbf{w}}_i - \nabla_{\hat{\mathbf{w}}_i} J(\hat{\mathbf{w}}_i), \quad (9)$$

where $\nabla_{\hat{\mathbf{w}}_i} J(\hat{\mathbf{w}}_i)$ denotes the gradient of $J(\hat{\mathbf{w}}_i)$ with respect to $\hat{\mathbf{w}}_i$ as follows:

$$\nabla_{\hat{\mathbf{w}}_i} J(\hat{\mathbf{w}}_i) = \begin{cases} -s_1 \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|^2} e_i, & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} \leq \sqrt{\delta_i} \\ -s_2 \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} \text{sgn}(e_i), & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} > \sqrt{\delta_i} \end{cases}, \quad (10)$$

where

The $\text{sgn}(\bullet)$ denotes the sign function. Fig. 1 shows the proposed cost function $J(\hat{\mathbf{w}}_i)$ for various δ 's.

$$s_1 \triangleq \frac{4 \exp\left(-\left(e_i / \|\mathbf{u}_i\|\right)^2\right)}{\left(1 + \exp\left(-\left(e_i / \|\mathbf{u}_i\|\right)^2\right)\right)^2},$$

$$s_2 \triangleq \frac{4 \exp\left(-2\left(\sqrt{\delta_i} \frac{|e_i|}{\|\mathbf{u}_i\|} - \frac{\delta_i}{2}\right)\right)}{\left(1 + \exp\left(-2\left(\sqrt{\delta_i} \frac{|e_i|}{\|\mathbf{u}_i\|} - \frac{\delta_i}{2}\right)\right)\right)^2}. \quad (11)$$

The proposed algorithm has two-type: an NLMS with the step size $\mu = s_1$ or the NLMAD with the step size $\mu = s_2$. However, we can insist that the proposed algorithm is VSS NLMS algorithm with step size

$$\mu_i = \begin{cases} \frac{4 \exp\left(-\left(e_i / \|\mathbf{u}_i\|\right)^2\right)}{\left(1 + \exp\left(-\left(e_i / \|\mathbf{u}_i\|\right)^2\right)\right)^2}, & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} \leq \sqrt{\delta_i} \\ \frac{4 \exp\left(-2\left(\sqrt{\delta_i} \frac{|e_i|}{\|\mathbf{u}_i\|} - \frac{\delta_i}{2}\right)\right)}{\left(1 + \exp\left(-2\left(\sqrt{\delta_i} \frac{|e_i|}{\|\mathbf{u}_i\|} - \frac{\delta_i}{2}\right)\right)\right)^2} \frac{\|\mathbf{u}_i\|}{|e_i|}, & \text{if } \frac{|e_i|}{\|\mathbf{u}_i\|} > \sqrt{\delta_i}. \end{cases} \quad (12)$$

III. SIMULATION

In the simulation of channel identification, the order of the adaptive filter is 32, which is equal to that of the corresponding unknown channel, and $\mathbf{w}$ is randomly generated as a unit vector. The input sequence is an AR1 with pole in 0.85 and simulation results are obtained from 100 independent trials. The impulsive measurement noise η_i is added to $y_i = \mathbf{u}_i^T \mathbf{w}$ with 30dB signal-to-noise ratio (SNR) and the SNR is calculated by

$$\text{SNR} = 10 \log_{10} \left(\frac{E(y_i^2)}{E(v_i^2)} \right). \quad (13)$$

The parameter α , the update equation of δ_i , and η_i are assigned as in [11]. η_i is generated as $\eta_i = \omega_i N_i$, where ω_i is Bernoulli process with $\Pr(\omega_i=1)=p$ and N_i is zero-mean Gaussian with power. $\sigma_N^2 = 1000\sigma_y^2$. For comparison, we use the NLMAD [5] and the robust VSS (RVSS) NLMS [11] algorithm.

First, we study when impulsive measurement noise does not appear. In Fig. 2, the performance of the proposed algorithm is similar to that of the RVSS NLMS algorithm. Next, we consider the impulsive measurement noise under 2 cases: $p=0.1$ and 0.5 . In Fig. 3, the proposed algorithm leads to better than those of other compared robust adaptive algorithms. Moreover, Fig. 4 shows that the proposed algorithm leads to the robustness against impulsive noise of high probability in robust adaptive algorithms, although the probability is very inadequate condition in the signal

processing applications.

Fig. 2. MSD learning curves (no impulsive measurement noise, $p=0$).

Fig. 3. MSD learning curves with impulsive measurement noise ($p=0.1$).

Fig. 3. MSD learning curves with impulsive measurement noise ($p=0.5$).

IV. CONCLUSION

This letter introduces a new gradient-based adaptive filtering algorithm based on a cost function that is constructed by combining two robust cost functions, which are a new *tanh*-type cost function and Vega's cost function. Through the approach to combine robust cost functions, the robustness of the proposed algorithm outperforms that of other adaptive algorithms. Since the proposed algorithm is derived by combining two robust cost functions, it leads to an excellent transient and steady-state behavior in high probability of impulsive measurement noise. Simulations have shown the performance of the proposed algorithm is better than those of the other robust adaptive algorithms in various probability of impulsive measurement noise.

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