

Signal Interpolator Design Using Weighted-Least-Squares Method

Noboru Ito and Wei Qin

Abstract—This paper shows that the bandwidth of a cubic interpolator can be designed by using a weighted least-squares (WLS) method in the frequency-domain. We first show how to construct a length-6 cubic whose support is 6 by connecting three piecewise polynomials of the third degree, and then explain how to find its optimal coefficients through minimizing the weighted squared error between the ideal and actual frequency responses of the length-6 cubic. By adjusting the weighting functions on different frequency bands, we can easily control the bandwidth of interest. In practical applications, according to the distribution of the signal to be interpolated, we can adjust the weighting function in the interpolator design and thus obtain very accurate frequency bands. As a consequence, the original band-limited analog signal can be accurately reconstructed, which also leads to high-accuracy signal interpolation. A narrow-band design example is given to illustrate that the length-6 cubic can achieve much higher accuracy frequency response than other existing interpolators. Moreover, we also detail how to choose the weighting function including "don't care" frequency bands.

Index Terms—Signal processing, signal reconstruction, sinc function, re-sampling, signal interpolation, interpolator design.

I. INTRODUCTION

Signal resampling is a two-step process; the first step is to reconstruct the original band-limited analog signal from

The given discrete signal using an interpolator and the second step is to resample the constructed analog signal at a different sampling-rate. The ideal interpolation kernel is the sinc function $h(t)$ defined as

$$h_d(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad (1)$$

whose frequency response is an ideal low-pass filter

$$H_d(\omega) = \begin{cases} 1, & |\omega| \leq \pi \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

And w is the normalized angular frequency. In practice, this ideal low-pass characteristic cannot be exactly realized since the sinc function is spatially unlimited. Therefore, many interpolation kernels with finite-lengths have been developed for practical signal interpolations [1]-[6].

This paper proposes a frequency-domain weighted

least-squares (WLS) method for designing a length-6 cubic whose support length is 6. The length-6 cubic is symmetric and formed by connecting three piecewise polynomials of the third degree. The optimal coefficients are found by minimizing the WLS error between the desired and actual frequency responses subject to a few constraints. Those constraints are imposed such that the length-6 cubic keeps the original discrete signal values unchanged after signal interpolation. Also, the first-order derivative continuity is also imposed. Since the length-6 cubic has 12 free parameters (coefficients), after imposing the above constraints on the design formulation, the remaining 2 free parameters are optimized in the WLS error sense, which improves the accuracy of the resulting length-6 cubic. A narrow-band design example is given to illustrate the excellent performance of the length-6 cubic. This paper also details how to choose the weighting function according to the spectra of the discrete-time signals to be interpolated. By changing the weighting functions in different frequency bands, the interpolator can be designed to be signal dependent. Thus, this frequency-domain WLS design approach has high flexibility. This high flexibility in turn leads to high-accuracy signal-dependent interpolators.

II. INTERPOLATOR AND FREQUENCY RESPONSE

We define the time-domain length-6 cubic as

$$h(t) = \begin{cases} a_3 |t|^3 + a_2 |t|^2 + a_1 |t| + a_0, & |t| \leq 1 \\ b_3 |t|^3 + b_2 |t|^2 + b_1 |t| + b_0, & 1 \leq |t| \leq 2 \\ c_3 |t|^3 + c_2 |t|^2 + c_1 |t| + c_0, & 2 \leq |t| \leq 3 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where the coefficient vectors

$$\begin{aligned} a &= [a_3 \ a_2 \ a_1 \ a_0] \\ b &= [b_3 \ b_2 \ b_1 \ b_0] \\ c &= [c_3 \ c_2 \ c_1 \ c_0] \end{aligned}$$

are to be determined. Obviously, the interpolation kernel is even-symmetric and its support is 6. This is why we call it length-6 cubic. The frequency response of $h(t)$ can be computed by using the Fourier transform as

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = 2 \int_0^3 h(t) \cos(\omega t) dt \quad (4)$$

III. FREQUENCY-DOMAIN WLS DESIGN

To find the coefficient vectors a , b , and c , we impose the

Manuscript received August 17, 2012; revised September 14, 2012.

N. Ito is with the Department of Information Science, Faculty of Science, Toho University, Miyama 2-2-1, Funabashi, Chiba 274-8510, Japan (e-mail: 050488in@is.sci.toho-u.ac.jp).

W. Qin is with Dalian Polytechnic University, China.

following constraints on the cubic (3):

- 1) $h(t)=1$ for $t=0$;
- 2) $h(t)=0$ for $t=1,2,3$;
- 3) $h(t)$ is continuous at the contacting points $t=1,2,3$;
- 4) $h'(t)$ (the first-order derivative of $h(t)$) is continuous at the contacting points $t=0,1,2,3$.

By applying the above constraints to the interpolation kernel (3), we get the following 10 constraints

$$\left\{ \begin{array}{l} a_0=1 \\ a_3+a_2+a_1+a_0=0 \\ b_3+b_2+b_1+b_0=0 \\ 8b_3+4b_2+2b_1+b_0=0 \\ 8c_3+4c_2+2c_1+c_0=0 \\ 27c_3+9c_2+3c_1+c_0=0 \\ a_1=0 \\ 3a_3+2a_2+a_1=3b_3+2b_2+b_1 \\ 12b_3+4b_2+b_1=12c_3+4c_2+c_1 \\ 27c_3+6c_2+c_1=0. \end{array} \right.$$

Since the length-6 cubic (3) has 12 coefficients, and the above 10 constraints utilize 10 coefficients, the number of free parameters (coefficients) becomes 2. If we take b_3 and b_2 as free parameters, the remaining coefficients can be expressed as

$$\left\{ \begin{array}{l} a_3 = -4 b_3 - b_2 + 2 \\ a_2 = 4 b_3 + b_2 - 3 \\ a_1 = 0 \\ a_0 = 1 \\ b_1 = -7 b_3 - 3 b_2 \\ b_0 = 6 b_3 + 2 b_2 \\ c_3 = 5 b_3 + b_2 \\ c_2 = -4 b_3 - 8 b_2 \\ c_1 = 10 b_3 + 2 b_2 \\ c_0 = -9 b_3 - 18 b_2. \end{array} \right.$$

As a result, the frequency response can be compactly expressed as

$$H(\omega) = b_2 f(\omega) + b_3 g(\omega) + z(\omega)$$

with

$$f(\omega) = \frac{12 \sin(\omega) - 12 \sin(2\omega) - 4 \sin(3\omega)}{\omega^3} + \frac{12 \cos(\omega) + 12 \cos(2\omega) - 12 \cos(3\omega) - 12}{\omega^4} \quad (6)$$

$$g(\omega) = \frac{44 \sin(\omega) - 64 \sin(2\omega) - 20 \sin(3\omega)}{\omega^3} + \frac{60 \cos(\omega) + 48 \cos(2\omega) - 60 \cos(3\omega) - 48}{\omega^4} \quad (7)$$

$$z(\omega) = \frac{-12 \sin(\omega)}{\omega^3} + \frac{24 - 24 \cos(\omega)}{\omega^4}.$$

Thus, the weighted squared error of the frequency response is

$$\begin{aligned} e(b_2, b_3) &= \int_{-\infty}^{\infty} W(\omega) [H(\omega) - H_d(\omega)]^2 d\omega \\ &= \int_{-\infty}^{\infty} W(\omega) [b_2 f(\omega) + b_3 g(\omega) + z(\omega) - H_d(\omega)]^2 d\omega \end{aligned} \quad (8)$$

where the weighting function $W(\omega)$ is selected as

$$W(\omega) = \begin{cases} 1, & \omega \in (2k\pi - \alpha\pi, 2k\pi + \alpha\pi), k = 0, \pm 1, \pm 2, \pm 3 \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

The reason why the weighting function $W(\omega)$ is chosen as such will be detailed later. To find the optimal coefficients b_2 and b_3 , we differentiate $e(b_2, b_3)$ with respect to b_2 and b_3 , respectively, and then set the derivatives to zero, thus

$$\begin{aligned} & b_2 \underbrace{\int_{-\infty}^{\infty} W(\omega) f^2(\omega) d\omega}_{A_{11}} + b_3 \underbrace{\int_{-\infty}^{\infty} W(\omega) f(\omega) g(\omega) d\omega}_{A_{12}} \\ &= \underbrace{\int_{-\infty}^{\infty} W(\omega) [H_d(\omega) - z(\omega)] f(\omega) d\omega}_{d_1} \\ & b_2 \underbrace{\int_{-\infty}^{\infty} W(\omega) f(\omega) g(\omega) d\omega}_{A_{21}} + b_3 \underbrace{\int_{-\infty}^{\infty} W(\omega) g^2(\omega) d\omega}_{A_{22}} \\ &= \underbrace{\int_{-\infty}^{\infty} W(\omega) [H_d(\omega) - z(\omega)] g(\omega) d\omega}_{d_2}. \end{aligned} \quad (10)$$

The above equations can be expressed in matrix form as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (11)$$

By solving the matrix equation (11), we yield the optimal coefficients b_2 and b_3 . Other remaining coefficients can be easily computed by using the relations in (5).

Next, let us consider why the weighting function is selected as (9). Assume that f_i are the sampled values of the band-limited analog signal $f(t)$. By utilizing the interpolation kernel $h(t)$ and the discrete-time signal values f_i , we can approximately reconstruct the analog signal as

$$\hat{f}(t) = \sum_{i=-\infty}^{\infty} f_i h(t - i)$$

which is called the convolution. Therefore, the frequency response of the reconstructed analog signal is

$$\hat{F}_i(\omega) = F_i(\omega) H(\omega) \quad (12)$$

where $F_i(\omega)$ and $H(\omega)$ are the Fourier transforms of f_i and $h(t)$, respectively, and ω is the normalized angular frequency. As shown in Fig. 1, if $f(t)$ is band-limited, then $F_i(\omega)$ is periodic as shown in Fig. 2. In the ideal case, if the frequency response of the designed interpolator is identical to the desired one, then the reconstructed analog signal has exactly the same frequency response as $f(t)$. This ideal reconstruction process is shown in Fig. 3. However, in practice, since the actual interpolator cannot be designed exactly equal to the desired one, then the reconstructed analog signal has

non-zero frequency response outside the frequency range $\omega \in [-\alpha\pi, \alpha\pi]$ as shown in Fig. 4, where some errors occur in the frequency bands $\omega \in [2k\pi - \alpha\pi, 2k\pi + \alpha\pi]$. Such errors cause aliasing problem when re-sampling the reconstructed analog signal. Therefore, the errors in the frequency bands $\omega \in [2k\pi - \alpha\pi, 2k\pi + \alpha\pi]$ should also be suppressed as shown in Fig. 5, which can be done by using the weighting function selected as (9). The remaining frequency bands can be regarded as "don't care" bands since no frequency components exist in such bands.

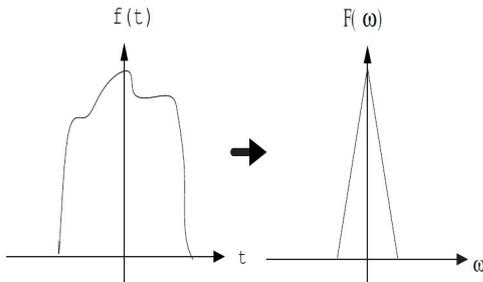


Fig. 1. Band-limited analog signal and its spectrum.

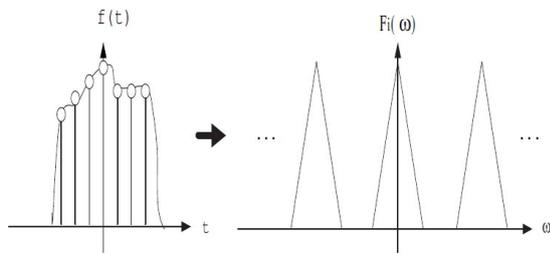


Fig. 2. Sampled signal and its spectrum

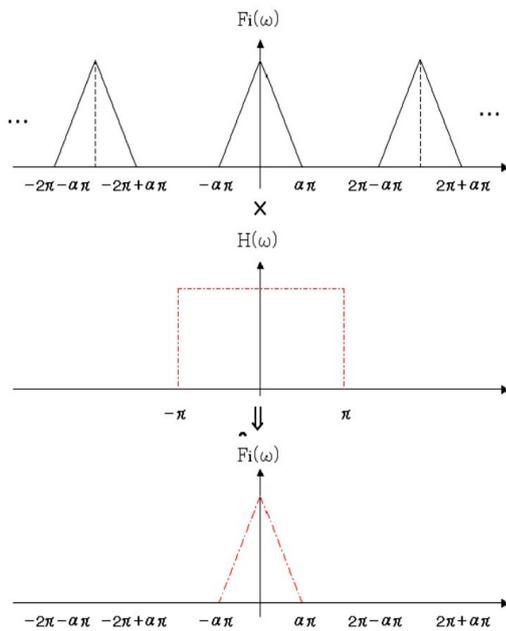


Fig. 3. Reconstruction using ideal interpolator

IV. NARROW-BAND EXAMPLE

Assume that the signal to be interpolated contains low-frequency components. For example, we set $\alpha=0.15$ and obtain the narrow-band length-6 cubic

$$h(t) = \begin{cases} 1.3191|t|^3 - 2.3191|t|^2 + 1 \\ -0.5902|t|^3 + 3.0419|t|^2 - 4.9938|t| + 2.5422 \\ 0.0905|t|^3 - 0.7247|t|^2 + 1.9025|t| - 1.6307 \end{cases} \quad (13)$$

Fig. 6 depicts the time-domain interpolation kernel, and Fig. 7 illustrates its frequency response. Table I tabulates the maximum absolute errors and normalized root-mean-square (NRMS) errors of frequency responses of various interpolators, where the errors are computed in the frequency bands

$$\omega \in [2k\pi - 0.15\pi, 2k\pi + 0.15\pi], (k=0, \pm 1, \pm 2, \pm 3)$$

Clearly, the designed narrow-band length-6 cubic has the smallest NRMS error and maximum error. That is, the resulting narrow-band cubic has the highest design accuracy.

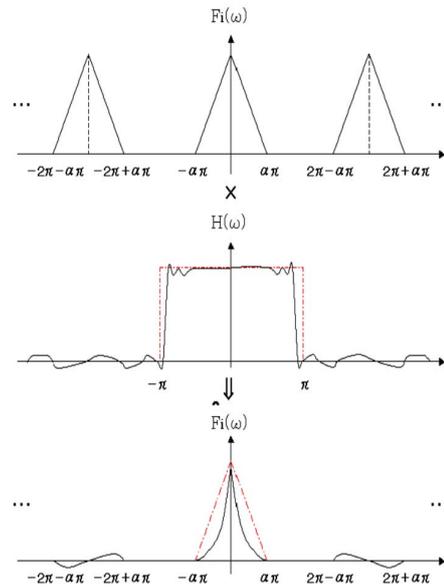


Fig. 4. Reconstruction using actual (non-ideal) interpolator

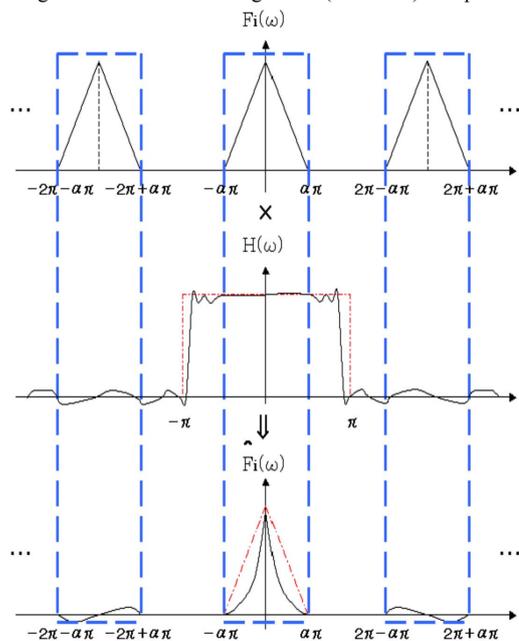


Fig. 5. WLS design using don't care bands.

Finally, let us provide some comments on the signal interpolation using variable fractional-delay (VFD) digital

filters [7]-[24]. VFD digital filters are a new type of digital filters that can be used directly for interpolating discrete-time signals. Recently, many efficient methods have been developed for designing VFD filters. But for the applications of only signal reconstruction, i.e., we only need to approximately reconstruct the original analog signal, the VFD filters cannot be used because they operate on discrete-time signals and then produce also discrete-time signals. Thus, for the reconstruction of an analog signal, interpolation kernels (analog low-pass filters) like the length-6 cubic proposed in this paper must be used. Moreover, once the analog signal is reconstructed by using the interpolation kernel, new discrete-time signal can be easily obtained by re-sampling the reconstructed analog signal, which has been mentioned in Introduction. As a result, the continuous-time interpolation kernels cannot only be used in the reconstruction of the original analog signal, but also can be applied to signal interpolation, but the signal interpolation requires two steps (reconstruction and re-sampling).

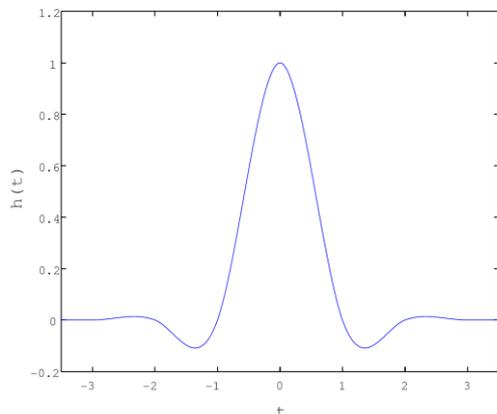


Fig. 6. Narrow-band length-6 cubic

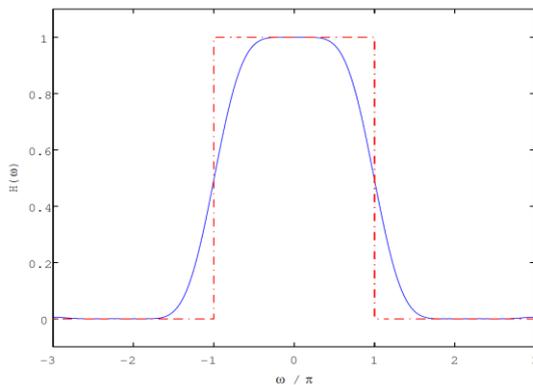


Fig. 7. Frequency response of narrow-band cubic

TABLE I: FREQUENCY RESPONSE ERRORS $\alpha=0.15$

Interpolation Kernel	Maximum Error	NRMS Error (%)
Nearest Neighbour	0.07976	7.1216
Linear	0.01836	0.9079
Quadratic	0.00936	0.5649
Catmull-Rom Cubic	0.00110	0.0514
Narrow-Band Cubic	0.00006	0.0024

V. CONCLUSION

Since the bandwidth of the length-6 cubic can be adjusted through adjusting the weighting functions, different cubic interpolators can be designed for interpolating different

signals with different spectra. As compared with the existing interpolators, the WLS frequency-domain cubic interpolators can be designed with changeable bandwidth. Thus, the new design approach has higher flexibility than the existing ones.

As future research topic, it is necessary to develop new frequency-domain methods for designing interpolators with maximum frequency response error (peak error) suppressed.

ACKNOWLEDGEMENT

Professor N. Ito would like to thank Mr. Taisaku Ishiwata (his former master student) for helping the authors draw Figs. 1-5 that explain the principle of the selected weighting function $W(w)$ in (9).

REFERENCES

- [1] N. A. Dodgson, "Quadratic interpolation for image resampling," *IEEE Trans. Image Process.*, vol. 6, no. 9, pp. 1322-1326, Sept. 1997.
- [2] H. S. Hou and H. C. Andrews, "Cubic splines for image interpolation and digital filtering," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-26, no. 6, pp. 508-517, Dec. 1978.
- [3] R. G. Keys, "Cubic convolution interpolation for digital image processing," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-29, no.6, pp. 1153-1160, Dec. 1981.
- [4] T. B. Deng, "Frequency-domain weighted-least-squares design of signaldependent quadratic interpolators," *IET Signal Processing*, vol. 4, no. 1, pp. 102-111, Feb. 2010.
- [5] T. B. Deng, "Wide-band length-6 cubic interpolator," in *Proc. of 2010 IEEE Int. Symp. Circuits Syst.*, pp. 445-448, Paris, France, May 30-Jun. 2, 2010.
- [6] T. B. Deng, "High-accuracy length-6 cubic interpolators," *IEEE Trans.Signal Process. Lett.*, vol. 17, no. 11, pp. 925-928, Nov. 2010.
- [7] T. B. Deng, "Discretization-free design of variable fractional-delay FIR digital filters," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 48, no. 6, pp. 637-644, Jun. 2001.
- [8] T. B. Deng and Y. Lian, "Weighted-least-squares design of variable fractional-delay FIR filters using coefficient-symmetry," *IEEE Trans. Signal Process.*, vol. 54, no. 8, pp. 3023-3038, Aug. 2006.
- [9] T. B. Deng, "Hybrid structures for low-complexity variable fractional-delay FIR filter," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 4, pp. 897-910, Apr. 2010.
- [10] T. B. Deng, "Closed-form design and efficient implementation of variable digital filters with simultaneously tunable magnitude and fractional-delay," *IEEE Trans. Signal Process.*, vol. 52, no. 6, pp. 1668-1681, Jun. 2004.
- [11] T.-B. Deng and Y. Nakagawa, "SVD-based design and new structures for variable fractional-delay digital filters," *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2513-2527, Sep. 2004.
- [12] T. B. Deng, "Design of complex-coefficient variable digital filters using successive vector-array decomposition," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 5, pp. 932-942, May 2005.
- [13] T. B. Deng, "Coefficient-symmetries for implementing arbitrary-order Lagrange-type variable fractional-delay digital filters," *IEEE Trans. Signal Process.*, vol. 55, no. 8, pp. 4078-4090, Aug. 2007.
- [14] T. B. Deng, "Symmetric structures for odd-order maximally flat and weighted-least-squares variable fractional-delay filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 12, pp. 2718-2732, Dec. 2007.
- [15] T. B. Deng, "Robust structure transformation for causal Lagrangetype variable fractional-delay filters," *IEEE Trans.*

Circuits Syst. I: Reg. Papers, vol. 56, no. 8, pp. 1681-1688, Aug. 2009.

- [16] J. J. Shyu and S. C. Pei, "A generalized approach to the design of variable fractional-delay FIR digital filters," *Signal Process.*, vol. 88, no. 6, pp. 1428-1435, Jun. 2008.
- [17] T. B. Deng, "Minimax design of low-complexity even-order variable fractional-delay filters using second-order cone programming," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 58, no. 10, pp. 692-696, Oct. 2011.
- [18] T. B. Deng, "Decoupling minimax design of low-complexity variable fractional-delay FIR digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 10, pp. 2398-2408, Oct. 2011.
- [19] C. C. Tseng, "Design of 1-D and 2-D variable fractional delay allpass filters using weighted least-squares method," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 10, pp. 1413-1422, Oct. 2002.
- [20] T. B. Deng, "Noniterative WLS design of allpass variable fractional-delay digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, no. 2, pp. 358-371, Feb. 2006.
- [21] T. B. Deng, "Generalized WLS method for designing allpass variable fractional-delay digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 56, no. 10, pp. 2207-2220, Oct. 2009.
- [22] T. B. Deng, "Minimax design of low-complexity allpass variable fractional-delay digital filters," *IEEE Trans. Circuits Syst. I: Reg. Papers*, vol. 57, no. 8, pp. 2075-2086, Aug. 2010.
- [23] T. B. Deng, "Closed-form mixed design of high-accuracy all-pass variable fractional-delay digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 5, pp. 1008-1019, May 2011.

- [24] T. B. Deng, S. Chivapreecha, and K. Dejhan, "Bi-minimax design of even-order variable fractional-delay FIR digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 59, no. 8, pp. 1766-1774, Aug. 2012.



Noboru Ito received the Ph.D. degree in electronic engineering from Tohoku University, Sendai, Japan, in 1991. He is a Professor with the Department of Information Science, Faculty of Science, Toho University, Funabashi, Japan. In 1992, he was selected by the Japanese Government as a Special Researcher for carrying out the Basic-Science-Program at the Institute of Physical and Chemical Research (RIKEN), Wako, Japan. His research interests focus on the design and implementation of variable one-dimensional (1-D) and variable multi-dimensional (M-D) digital filters.

He served as an Editorial Board member of Elsevier's Signal Processing from Apr. 2009 to Mar. 2012, and was a member of IEICE CAS Technical Committee (Japan) from May 2009 to Mar. 2012. Currently, Dr. Ito is a member of Digital Signal Processing (DSP) Technical Committee of the IEEE Circuits and Systems (CAS) Society, and an Associate Editor (AE) for IEEE Trans. Circuits and Systems II: Express Briefs.



Wei Qin was born in Liaoning, China, in 1962. He received the Bachelor's degree in automatic control engineering and Master's degree in machinery design and theory from Dalian Polytechnic University, China, in 1984 and 2004, respectively. At present, he is an Associate Professor with the same university. His research interests include automatic control theory, digital communication, and digital signal processing.