Control of Linear Stochastic Systems with Constraints Asymptotic Predictive Method (APM)

Enes Saletović and Tadej Mateljan

***Abstract*—The scope of this work is research of the problem linear stochastic control systems with constraints. Motivation to solving this problem derives from the fact that there is no universal solution to this problem, even though stochastic systems with constraints are very common in practice. Defined control problem is solved by creating new predictive control method. Validity and convergence of the created method is tested by method of simulation at many different scenarios of constraints and stochastic disturbances.**

***Index Terms*—Control, stochastic, constraints, prediction.**

I. Introduction

In theory, most of the real time systems by nature are stochastic, because of ever present disturbances and noises at different system variables. Because of actuators constraints, all real time systems contain constraints at least in input variables, and often in other system variables too. For instance, human being functions as a very complex stochastic system with numerous constraints. If influence of disturbances and measurement noises to system dynamics is negligible, system variables are treated as deterministic values, therefore such system is considered deterministic. Systems with at least one variable or system parameter as stohastic value are considered stochastic. Such systems' dynamics could not be described as deterministic and stochastic component influence is not negligible [1]. Disturbance as stochastic (unpredictable) value may be present at one or more system variables, either at input or output. Influence of measurement noise to the system dynamics is usually lot less effective then disturbance influence. Developed deterministic methods of control are not directly applicable to stochastic systems with constraints. For instance, system of automatic control with negative feedback successfully removes disturbance influence, but could not meet system constraints. Further point of interest will be the problem of control in linear stochastic systems with constraints, with negligible small measurement noise. It is assumed that all needed conditional variables are measurable or their values could be determined by some of estimation methods. Stohastic systems could be continuous, discreet or hybrid. For instance, system of control in hydro accumulations is continuous, while system of control supply goods is discreet stochastic system. The goal of control is always the same: control the system in accordance with given criteria of control with presence of disturbances and noises, with respect to given constraints. Control is more qualitative as the influence of disturbance is more rejected or compensated.

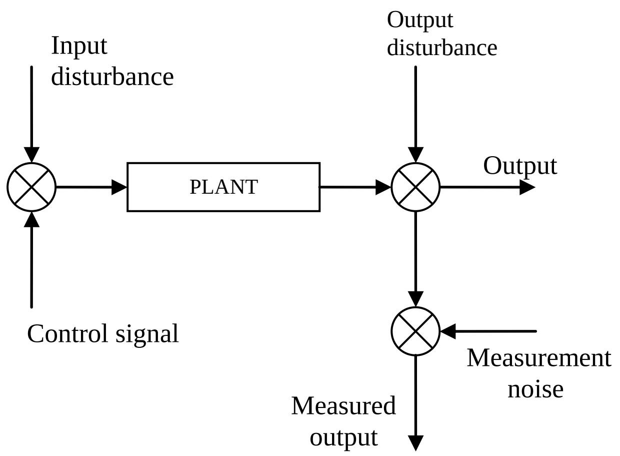

Fig. 1. Disturbance influence in stochastic system.

II. System Constraints [2]

Constraints are always present at input variables and could be present at state variables as well as output variables. For instance, regulation valve as an actuator has constructional constraints in regard to maximal allowed flow. In general, box-like constraints appear at input variables as:

$$U_{\min} \le u \le U_{\max} \tag{1}$$

Because of own constraints of actuators dynamics and fluid dynamics, constraints at change rate variables appear as:

$$\Delta U_{\min} \le \Delta u \le \Delta U_{\max} \tag{2}$$

Constraints at output variables and state variables could be qualitative and quantitative. Qualitative constraints usually do not emerge at input system variables. Constraints that could not be violated are called hard constraints. For instance, maximum temperature of flammable fluid are hard constraints. Constraints that could be violated with certain deviation to control quality are called soft constraints (for instance, reject < *5%*).

III. Mathematical Problem Formulation

The problem of control in linear stochastic system is usually formulated in discreet domain, in state space as:

$$x(k+1) = Ax(k) + Bu(k) + Ew(k) \tag{3}$$

Manuscript received April 28, 2012; revised June 6, 2012.
Enes Saletović and Tadej Mateljan are with the University of Sarajevo.

$$y(k) = Cx(k) + Du \quad (4)$$

where:
$x(k)$ – State variables vector
$u(k)$ – Input variables vector
$y(k)$ – Output variables vector
$w(k)$ – System disturbances vector
A, B, C, D, E – Appropriate system matrix [3]

Analysis and synthesis of automatic control system in stochastic systems is done within discreet domain, because the goal is control algorithm applicable to software implementation. Control goal is minimizing the cost function $J(k)$ used to grade control quality. It is necessary at the beginning of each discreet time interval ($t=kT$) to generate optimal vector of control signals $u^*(k)$.

$$u^*(k) = \arg\min_u J(k) \quad (5)$$

Because in every moment ($t=kT$) stands [1]:

$$u^*(k) = u(k-1) + \Delta u^*(k) \quad (6)$$

derives that at the beginning of each discreet time interval optimal correction is calculated $\Delta u^*(k)$ because vector $u(k-1)$ at given moment ($t=kT$) is known. It is assumed that computing time $u^*(k)$ is a lot shorter then sampling time T. Stochastic disturbance of normal distribution are assumed $f(w)$ as in Figure (2).

IV. Model Predictive Control (MPC)

Model predictive control (MPC) means knowing mathematical model of the system under control. It is assumed that all necessary variables of the system are measurable or estimated. Established mathematical model is used in concept MPC to predict behavior of controlled system to a finite number of discrete time period T in advance. Time period H_p system behavior is predicted for is called prediction horizon. Within the frame of predictive control algorithm, appropriate cost function $J(k)$ is defined which most often means minimizing deviation of controlled variable in relation to referent model response, with minimal variation of control signal. The task of predictive control is to find a sequence of control signals in the predictive control horizon H_u which will minimize established cost function in respect to present constraints. To the system is applied only first control signal, after which new variable system measurement are taken for next iteration of computing optimal control signal. Computing optimal control sequence $u^*(k)$ is repeated in succession

$$\hat{u}^*(k) = \arg\min_u J(k) \quad (7)$$

Control horizon remains the same but with each new step it moves one step to the right (moving horizon). For one input and one output, cost function criteria J of standard predictive GPC (General Predictive Control) method has this form:

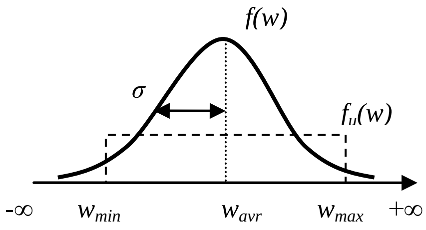

Fig. 2. Stochastic disturbance with normal $f(w)$ and limited uniformed $f_u(w)$ distribution.

$$J(k) = \sum_{j=H_w}^{H_p} \left[r(k+j) - \hat{y}(k+j)\right]^2 + \sum_{j=1}^{H_u} \Delta\hat{u}_j^2 \quad (8)$$

where:
$r(k)$ – Referential value
$\hat{y}(k)$ – Prediction vector of output values
$\Delta\hat{u}_j$ – Vector (series) of input correction
H_w – First prediction horizon (delay)
H_p – Prediction horizon
H_u – Control horizon

Idea of predictive control in its basis is simple but algorithm implementation is processor demanding, because of the minimizing cost function J complexity. Vector $\hat{y}$ is calculated by known mathematical model of the observed system.

$$\hat{y}(k+j) = f(\hat{u}(k+j-1), \hat{y}(k+j-1)), \quad \forall j \geq 1 \quad (9)$$

Referential value r is calculated as first order system response. Standard predictive control has several flaws which limit application in stochastic systems:

- Very complex matrix estimation of the optimal control signal $u^*(k)$
- Algorithm is software demanding
- Algorithm is hardware demanding
- Sometimes the problem of insolvability appears because of mathematical model imprecision

Most flaws of the GPC algorithm comes from the fact that in each time interval k calculates H_p optimal control signal, though only first computed signal is applied!

Below follows new MPC method for stohastic systems with constraints.

V. Asymptotic Predictive Method (APM)

For simplicity, concept of APM method is presented on one SISO (Single Input Single Output) system.

A. APM method idea

Idea of APM method is that on every time interval ($t=kT$) only first optimal control signal $u^*(k)$ is calculated using cost function $J(k)$, and all other prediction control signals in prediction horizon ($H_u=H_p$), are calculated based on $u(k)$ as:

$$u(k+j) = u(k) + \Delta u^+(k) \quad (10)$$

In respect to constraints:

$$u_{\min} \le u(k+j) \le u_{\max} \qquad \forall j \ge 1 \tag{11}$$

Counter control correction Δu^+ is:

$$\Delta u_{\min} \le \Delta u^+(k) \le \Delta u_{\max} \tag{12}$$

Counter control correction presents the worst possible control correction in regard to control error, it computes as follows:

$$\Delta u^+(k) = u^+(k) - u(k-1) \tag{13}$$

where the counter control signal is:

$$u^+(k) = \arg\max_u \left(\sum_{i=Hw}^{Hp} (r_i - \hat{y}_i)^2 \right) \tag{14}$$

Quadratic forms have maximum at domain border. Becouse of that, optimization problem (14) is very simple.

Counter control correction $\Delta u^+(k)$ becomes the correction which on observed control horizon H_u generates the largest control error. Cost function criteria of the APM method is:

$$J(k) = \sum_{j=H_w}^{H_p} \left[r - \hat{y}(k+j|k) \right]^2 + \hat{P}^z + \hat{P}^o + \hat{P}^w \tag{15}$$

where:

$\hat{y}(k+j|k)$ -Output prediction at the moment *kT*

r – Given set point value (constant)
H_w– First prediction horizon (delay)
H_p – Prediction horizon
P^z – Penalty of skipping set point values
P^o – Penalty of violation constraints
P^w – Penalty of violation constrains for disturbance

Every possible control signal $u(k)$ has unique sequence (10) from (H_u-1) counter control signals, one response trajectory S respectively. Diagram shows four concurrent prediction trajectories which represent prediction responses to four different control sequences. Trajectory S_1 is according to goal function criteria $J(k)$ (15) the worst because it violates constraint Y_{max}, on a observed prediction horizon, generates skipping set point value Y_{sp} penalties and significant error control. Trajectory S_2 does not violates above constraint Y_{max} but generates skipping set point value penalties P^z. Trajectory S_4 does not violates constraints but generates significant control error. Minimum cost function criteria $J(k)$ is generated by trajectory S_3 which represents system response to u^*- prediction control sequence, because on the observed prediction horizon does not generates penalties with minimum control error. Conditionally optimal control signal $u^*(k) \neq u_{op}(k)$ is defined numerically, by searching through possible control signals with satisfying control correctness.

$$u_{opt} - \varepsilon \le u^*(k) \le u_{opt} + \varepsilon \tag{16}$$

$$u(k-1) - \Delta U_{\max} \le u^*(k) \le u(k-1) + \Delta U_{\max} \tag{17}$$

Number of searches N is defined as:

$$N = \frac{2\Delta U_{\max}}{\varepsilon} + 1 \qquad 0 \prec \varepsilon \in R \tag{18}$$

where is arbitrary small positive real number.

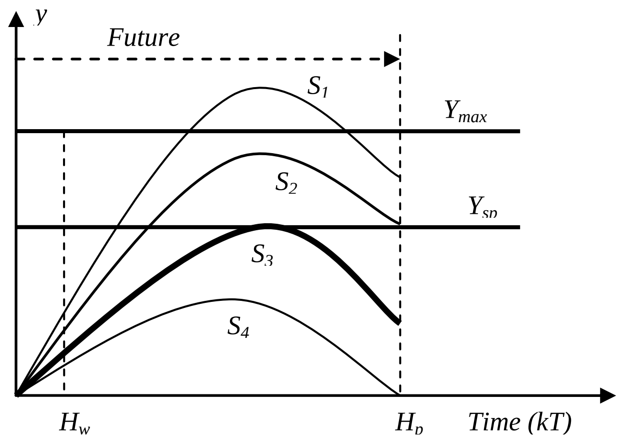

Fig. 3. Working principle of APM method.

B. Adjusting APM method parameters

Skipping set point value penalty P^z favors asymptotic approach to set point value and indirectly minimize control correction $\Delta u(k)$. Also, penalty P^z insures convergence of APM method. Depending on relation between set point and predicted output value in a moment k, penalty P^z is calculated as:

$$P^z = C^z \sum_{j=Hw}^{Hp} (r - \hat{y}_j)^2 \left[1 + \lambda \cdot Sign(\hat{y}_j - r) \right] \tag{19}$$

$$\lambda = \begin{cases} -1, & y_k \prec r \\ 1, & y_k \ge r \end{cases} \tag{20}$$

Weighted coefficient C^z adopted as:

$$10^3 Y_{\max} \le C^z \le 10^5 Y_{\max} \tag{21}$$

Violation constraint penalty P^o provides drastic punishment of violating present constraints. Therefore APM favors control signal which in prediction horizon respect constraints Y_{max} and asymptotic achievement of set point value r.

$$P^o = C^o \sum_{j=H_w}^{Hp} \left(\hat{y}_j - Y_m \right)^2 \left[1 + Sign(\hat{y}_j - Y_m) \right] \tag{22}$$

Weighted coefficient C^o adopted as:

$$10C^z \le C^o \le 10^2 C^z \tag{23}$$

Violation constraint penalty P^w due to maximum possible disturbances is calculated as:

$$P^w = C^w \approx 10C^o \ if \quad \exists i \in [1, H_u - 1] \quad y_i^w \succ Y_{\max} \quad (24)$$

if predicted output constraint violation y^w is foreseen while maximum possible disturbance w is present. Otherwise, ($P^w=0$) is valid. Sampling period is chosen from time constant τ of the observed system as:

$$\tau / 10 \leq T \leq \tau / 6 \quad (25)$$

APM method parameters are adjusted within the scope of automatic system control simulation.

C. Disturbance influence compensation

Disturbance influence in stochastic systems is usually reduced to input. Thus the disturbance is directly introduced to the mathematical model so it is possible to verify predicted system behavior even in the case of maximum possible stochastic disturbance.

Within APM method, disturbance influence is given as maximum possible predicted disturbance at input. Therefore APM method potentiates control of stochastic systems with respect to constraints even with the presence of maximum possible predicted disturbance. If controlled variable is far enough from marginal values, given value is reached asymptotically. However, if set point value is near to output constraints, APM method holds the values of controlled variable far enough from marginal values that not even eventual maximal disturbance could not cause violation constraints, as in Fig. 5. Depending upon adopted maximal disturbance w_{max} two approaches are possible: pessimistic and optimistic. Pessimistic approach is adopting higher values w_{max} to respect constraints and in case of unlikely, extreme disturbances. Optimistic approach is adopting lower values w_{max} to insure respect to present constraints in case of the most expected but not extreme disturbances. Pessimistic approach is stiff and often unacceptable because it endangers control correctness. Value w_{max} is adopted upon distribution diagram of expected disturbance w. Voluminous simulation controls based on APM method confirmed convergence and validity of created solutions.

Example 1:

Given control task: positioning the vehicle while limited stochastic disturbances are imposed (wind) w and friction F_t as in Figure (4). Limitations on input $u(k)$ and output $x(k)$ are in effect:

$$-1 \leq u(k) \leq 1 \quad (26)$$

$$-1 \leq \Delta u(k) \leq 1 \quad (27)$$

$$0 \leq x(k) \leq 10 \quad (28)$$

Mathematical model of the system is:

$$\ddot{x} + \frac{\mu}{m}\dot{x} = \frac{1}{m}u + \frac{1}{m}w \quad (29)$$

where:

x – Vehicle position (output – controlled variable)
u - Input (force as control variable)
m – Vehicle mass ($m=1$ *kg*)
μ - Friction coefficient ($\mu=1$ *kg/s*)
w- Normal distribution stochastic disturbance

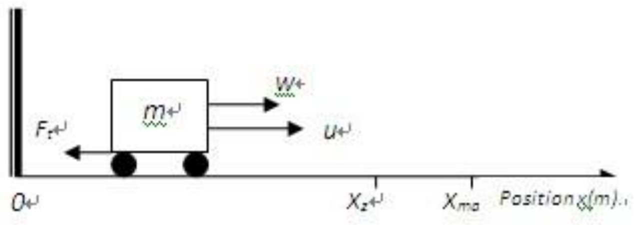

Fig. 4. Vehicle position control.

Acquired second order differential equation could be transformed to the system of two first order differential equation in state space. After exchange: $v = \dot{x}$ derives:

$$\dot{x} = v \quad (30)$$

$$\dot{v} = -\frac{\mu}{m}x_2 + \frac{1}{m}u + \frac{1}{m}w \quad (31)$$

Using adopted values ($m=\mu=1$), and sampling time (T=0.5s) acquired system of differential equations could be translated to discreet domain as a system of two differential equations in state space:

(32)

$$x(k+1) = x(k) + 0.5v(k)$$
$$v(k+1) = 0.5v(k) + 0.5u(k) + 0.5w(k)$$

with given the maximal disturbance (w_{maxx}=0.5 U_{max}).

Table I shows partial control simulation results of observed stochastic system. Fig. 5 shows dynamic behavior of the system in 20 s interval with variable set point position r. Though intensive stochastic disturbance of the maximal amplitude w_{max} is present, controlled output variable remains within given range.

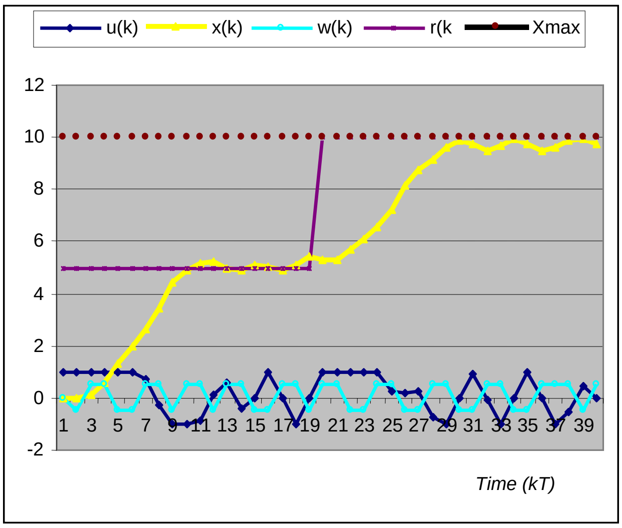

Fig. 5. Positioning the vehicle with present disturbance by applying APM method.

TABLE I: APM METHOD SIMULATION RESULTS

k	*u(k)*	*v(k)*	*x(k)*	*w(k)*	*r(k)*
1	1,00	0,00	0,00	0,00	5
2	1,00	0,25	0,00	-0,50	5
3	1,00	0,94	0,13	0,50	5
4	1,00	1,45	0,59	0,50	5
5	1,00	1,34	1,32	-0,50	5
6	1,00	1,25	1,99	-0,50	5
7	0,74	1,69	2,62	0,50	5
8	-0,26	1,89	3,46	0,50	5
9	-1,00	1,04	4,41	-0,50	5
10	-1,00	0,53	4,93	0,50	5
11	-0,88	0,15	5,19	0,50	5
12	0,12	-0,58	5,26	-0,50	5
13	0,56	-0,13	4,97	0,50	5
14	-0,44	0,44	4,91	0,50	5
15	0,00	-0,14	5,13	-0,50	5
16	1,00	-0,36	5,05	-0,50	5
17	0,00	0,48	4,88	0,50	5
18	-1,00	0,61	5,12	0,50	5
19	0,00	-0,29	5,42	-0,50	5
20	1,00	0,03	5,28	0,50	10

VI. CONCLUSION

Advantages of APM method over GPC method are: simplicity and compactness, modest hardware and software demands, solvability and convergence.

Disadvantages of APM method are: limited application to linear systems (with or without delay time) and algorithm complexity in regard to control multivariable systems.

As most of the real time objects could be approximated as second order system (with or without delay time), APM method presents efficient tool to solving a control problem of linear stochastic and deterministic systems with or without constraints.

REFERENCES

[1] J. M. Macijeowski, *Predictive Control with Constraints*, Pearson Education Ltd, 2001.

[2] I. Batina, *Model predictive control for stochastic systems by randomized algorithms*, Eindhoven, 2004.

[3] D. Chatterjee, P. Hokayem, and Y. Lugeros, *Stohastic reciding horizon control with bounded control inputs: a vector space approach*, Automatic Control Laboratory, Zurich, 2009.