A New Method of Image Denoising Based on the Energy Method

Shuang Liang, Shuli Wang, and Yu Wang

***Abstract*—To sove the minimization problem of the energy method in image denoising, the diffusion process of the Euler-Lagrange could be decomposed to the diffusions along the tangential direction and normal direction. In order to denoising and retained the texture information of the real scene as possible, at the texture pixels, the diffusion along the tangential direction should be retained and the diffusion along the normal direction should be controlled. Therefore we propose a new denoising method with edge diffusion along the tangential direction directly. The performance of this method is tested by single images in this thesis.**

***Index Terms*—Image denoising, energy method, edge diffusion, tangential direction, normal direction.**

I. Introduction

As for images in different scenes and with different kinds of noise type, many models have been studied in image denoising in the previous researches [1-30]. In this paper, we study the problem of image denoising based on the energy method [16], [21], [26]. Denote the real image as $u: \Omega \subset R^2 \rightarrow R$ and the degraded image as u_0. Then the degradation model of a digital image is

$$u_0 = Ru + \eta, \tag{1}$$

where R is a linear operator inducing image degradation for objective reasons, and η refers to Gaussian White Noise in this paper.

Tikhonov and Arsenin proposed energy functional added a regular term

$$E(u) = \int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega |\nabla u|^2 dx, \tag{2}$$

where the first term refers to the difference between the restored image and the degraded image, the second term means the total gradient modulus of the restored image and λ is a weighting parameter[1]. In regularization energy functional, two basic condition of denoising have been concerned. Firstly, the basic structures between the restored image and the degraded image must be close enough. Secondly, the total gradient modulus of the restored image must be smaller than which of the degraded image. Based on the above two basic conditions, the energy model was established. As a result, the problem of denoising has changed to the minimization problem of energy model

$$\inf_u\{E(u), u \in W^{1,2}(\Omega)\}. \tag{3}$$

Furthermore, in Ref. [3] Aubert and Vese proposed energy functional added a smooth term

$$E(u) = \int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx. \tag{4}$$

In order to solve the minimization problem, Chambolle proposed the projection approach in Ref. [6] and Lions proposed the half-quadratic minimization approach in Ref. [9]. These two methods both have better performance. In fact, in the previous researches, the Euler-Lagrange equation was also mentioned. The solution of the minimization problem also satisfied the Euler-Lagrange equation

$$\begin{cases} R^*Ru - R^*u_0 - \lambda\Delta u = 0 & in\ \Omega \\ \frac{\partial u}{\partial N} = 0 & on\ \partial\Omega \end{cases}. \tag{5}$$

In Ref. [23], Perona and Marik noticed that the diffusion process of the Euler-Lagrange could be decomposed to the diffusions along the tangential direction and normal direction respectively, then proposed nonlinear diffusion partial differential equation

$$\frac{\partial u}{\partial t}(t,x) = c(|\nabla u|^2)u_{TT} + b(|\nabla u|^2)u_{NN}. \tag{6}$$

In order to weaken noise and retain the texture information of the real scene as possible, at the texture pixels, the diffusion along the tangential direction should be retained and the diffusion along the normal direction should be controlled. Based on the above analysis, we propose a new denoising method with edge diffusion along the tangential direction directly. The performance of this method is tested by single images in this thesis.

The rest of the paper is organized as follows. In Section II, the detail of the proposed denoising method is presented. The experiments results and comparisons are shown in Section III. Finally, the conclusion is given in Section IV.

II. The Proposed Method

Consider the energy functional added a smooth term

$$E(u) = \int_\Omega |u_0 - Ru|^2 dx + \lambda \int_\Omega \phi(|\nabla u|) dx. \tag{7}$$

Supposing $E(u)$ has a minimum at$u$, it must satisfy the

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Euler-Lagrange equation

$$R^*Ru - \lambda div(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u) = R^*u_0, \quad (8)$$

And it is a diffusion equation. For x with $\nabla u(x) \neq 0$, define the unit normal vector of x as $N(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ and the unit tangent vector of x as $T(x), |T(x)| = 1, T(x) \perp N(x)$. Take note of

$$u_{TT} = T^t \nabla^2 u T = \frac{1}{|\nabla u|^2}(u_{x_1}^2 u_{x_2x_2} + u_{x_2}^2 u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2}), \quad (9)$$

$$u_{NN} = N^t \nabla^2 u N = \frac{1}{|\nabla u|^2}(u_{x_1}^2 u_{x_1x_1} + u_{x_2}^2 u_{x_2x_2} + 2u_{x_1}u_{x_2}u_{x_1x_2}). \quad (10)$$

where $u_{x_1}u_{x_2}$ refer to the first partial derivatives and $u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}$ refer to the second partial derivatives. Then the Euler-Lagrange equation can be rewrite

$$R^*Ru - \lambda(\frac{\phi'(|\nabla u|)}{|\nabla u|}u_{TT} + \phi''(|\nabla u|)u_{NN}) = R^*u_0. \quad (11)$$

The diffusion process of the Euler-Lagrange is decomposed to the diffusions along the tangential direction and normal direction. In order to denoising and retained the texture information of the real scene as possible, at the texture pixels, the diffusion along the tangential direction should be retained and the diffusion along the normal direction should be controlled. To come true isotropy in smooth area and the diffusion along the tangential direction, the function $\phi(s)$should satisfy

$$\phi'(0) = 0, \phi''(0) > 0,$$
$$\lim_{s\to+\infty}\phi''(s) = \lim_{s\to+\infty}\frac{\phi(s)}{s} = 0, \lim_{s\to+\infty}\frac{\phi''(s)}{\phi(s)/s} \quad (12)$$

Based on the above analysis, we propose a new denoising method with edge diffusion along the tangential direction directly as described below:

1) Preprocessing. To smooth the degraded image I_{in} locally and denote the processed image as SI_{in}.

2) Calculate the gradient vector ∇SI_{in} and $\nabla^{\perp} SI_{in}$ perpendicular to ∇SI_{in} at each point (i,j) in SI_{in}. Obviously, the unit normal vector is $N(i,j) = \frac{\nabla SI_{in}(i,j)}{|\nabla SI_{in}(i,j)|}$ and the unit tangent vector $T(i,j) = \frac{\nabla^{\perp} SI_{in}(i,j)}{|\nabla^{\perp} SI_{in}(i,j)|}$.

3) At each point (i,j) in SI_{in}, calculate the projections of $\nabla^{\perp} SI_{in}$ on the eight directions in nine points neighborhood $\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}, k = 1,2,\cdots,8$, where

$$e_1 = (1,0), e_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$
$$e_3 = (0,1), e_4 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$
$$e_5 = (-1,0), e_6 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right),$$
$$e_7 = (0,-1), e_8 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right). \quad (13)$$

4) Diffusion. Assign different weights as the following method

$$SI_{in}(i+1,j) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_1}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i+1,j), \quad (14)$$

$$SI_{in}(i+1,j+1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_2}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i+1,j+1), \quad (15)$$

$$SI_{in}(i,j+1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_3}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i,j+1), \quad (16)$$

$$SI_{in}(i-1,j+1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_4}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i-1,j+1), \quad (17)$$

$$SI_{in}(i-1,j) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_5}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i-1,j), \quad (18)$$

$$SI_{in}(i-1,j-1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_6}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i-1,j-1), \quad (19)$$

$$SI_{in}(i,j-1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_7}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i,j-1), \quad (20)$$

$$SI_{in}(i+1,j-1) + \lambda SI_{in}(i,j) \cdot \frac{|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_{81}}|}{\sum_{k=1}^{8}|\nabla^{\perp} SI_{in} \cdot \overrightarrow{e_k}| + \varepsilon} \to SI_{in}(i+1,j-1), \quad (21)$$

where $\lambda > 0, 0 < \varepsilon \ll 1$.

5) The image gray value increase after the process of diffusion. So an appropriate method to make the gray value belonging in [0,255] is necessary. Here we use histogram matching method and average gray value translation method, thus keep the image average gray value and image histogram unchanged essentially.

6) Iteration. Repeat the above steps 1) - 5) if the image denoising result is unsatisfied.

III. EXPERIMENTAL RESULTS

To evaluate the performance of our method, we have tested it on some test images shown in Fig.1 - Fig.3. In the experiment, we add noise of Gaussian (0, 0.05) on each test image and observe denoising results of the first iteration and the next one applying our method. Furthermore, we measure the quality of the denoising images with some methods of image quality assessment, i.e. MSE, PSNR, SSIM, MSSIM, SNR, UQI, and VIFP, as is shown in Table I - Table III.

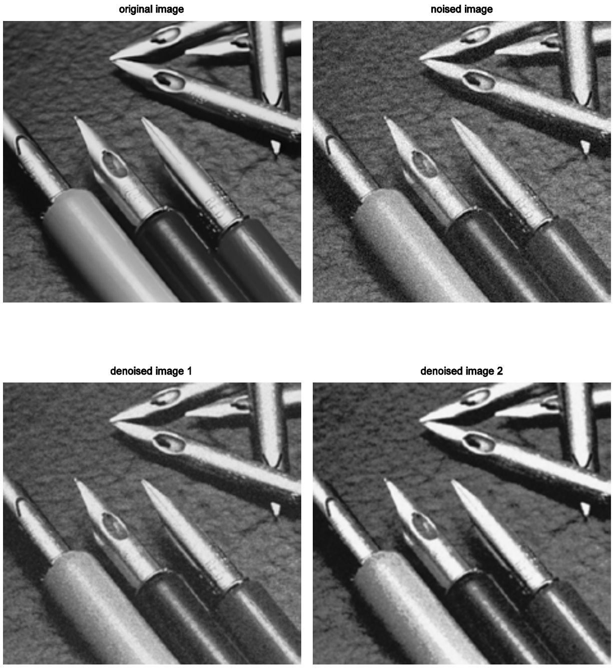

Fig. 1. Pens.

TABLE I: PENS

Methods of image quality assess-ment		Denoising results of the first iteration		Denoising results of the second iteration	
	noising image	denoisin g image	optimi -ed rate(%)	denoisin g image	optimi-ed rate(%)
MSE	781.3902	233.8726	70.07	232.1473	70.29
PSNR	19.2021	24.4410	27.28	24.4732	27.45
SSIM	0.3044	0.8347	174.18	0.8628	183.43
MSSIM	0.8194	0.9211	12.41	0.9221	12.53
SNR	11.7809	17.0198	44.47	17.0520	44.74
UQI	0.3212	0.6184	92.53	0.6532	103.37
VIFP	0.2232	0.3714	66.40	0.3893	74.41

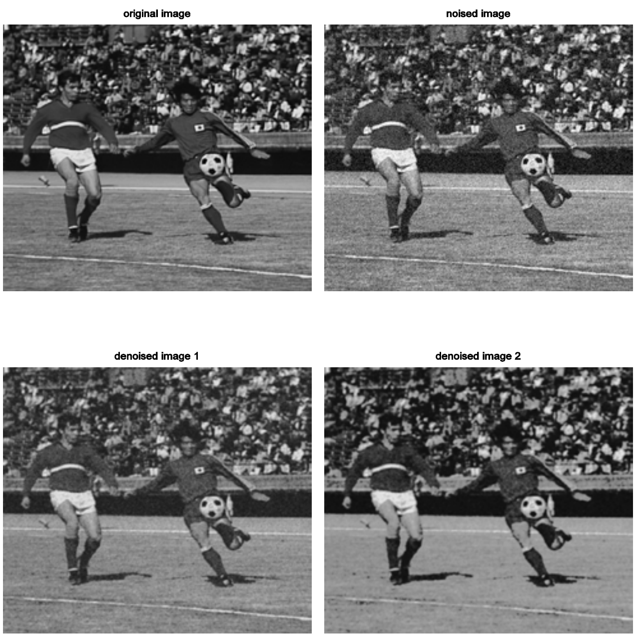

Fig. 2. Soccer.

TABLE II: SOCCER

Methods of image quality assess-ment		Denoising results of the first iteration		Denoising results of the second iteration	
	noising image	denoising image	optimiz-ed rate(%)	denoising image	optimiz-ed rate(%)
MSE	792.9716	294.0847	62.91	387.7826	51.10
PSNR	19.1382	23.4461	22.51	22.2449	16.23
SSIM	0.4337	0.8546	97.06	0.8350	92.54
MSSIM	0.8684	0.9282	6.89	0.8989	3.51
SNR	11.5662	15.8740	37.25	14.6729	26.86
UQI	0.4662	0.7009	50.33	0.6527	40.01
VIFP	0.2455	0.3771	53.64	0.3371	37.31

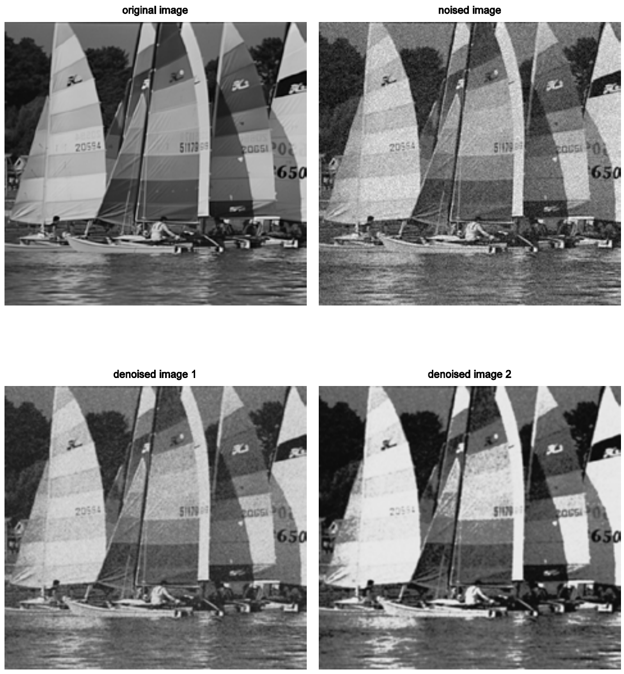

Fig. 3. Yach.

TABLE III: YACH

Methods of image quality assess-ment		denoising results of the first iteration		denoising results of the second iteration	
	noising image	denoising image	optimiz-ed rate(%)	denoising image	optimiz-ed rate(%)
MSE	797.4814	305.7834	61.66	727.1404	8.82
PSNR	19.1136	23.2767	21.78	19.5146	2.10
SSIM	0.3438	0.8190	138.24	0.8130	136.51
MSSIM	0.8126	0.9074	11.67	0.8746	7.63
SNR	12.5901	16.7531	33.07	12.9911	3.19
UQI	0.3449	0.5482	58.93	0.5297	53.58
VIFP	0.2242	0.3422	52.64	0.3285	46.53

IV. CONCLUSION

In this paper, we study the problem of image denoising based on the energy method. Based on the previous research, we propose a new denoising method with edge diffusion along the tangential direction directly. The performance of this method is tested by single images in this thesis.

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