Copulas and the Information Management

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***Abstract*—Copula is introduced as a tool for understanding the dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions, and provide a flexible way to describe nonlinear dependence; copulas therefore have been applied in many fields. Here we deal with a family of generalized Archimedean (GA) copulas. In terms of these GA copulas, we derive the formula for the Kendall's tau, which is a well known measure of concordance. Applications to the management are discussed.**

***Index Terms*—Copula, (generalized) Archimedean copula, Kendall's tau, information management.**

I. Introduction

Dependence relations between random events are one of the most important subjects for researches in probability and statistics. A copula function, or simply a copula, is introduced as a tool for understanding such a possibly nonlinear dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions. Because of their flexibilities, copulas have been applied in many situations from the financial risk management to the seismology, to name a few. The study of copulas thus has been very popular these days. We refer for instance to [1]-[7] and the references cited therin. One also finds an excellent monograph of R.B. Nelsen [8], as well as H. Tsukahara [9] and Y.Yoshizawa [10], [11] provide nice reviews. In addition, a well known book by A. J. McNeil, R. Frey, and P. Embrechts [12] contains the part of the theory of copulas.

In this note, we are concerned with the formula of the Kendall's tau for the generalized Archimedean (GA) copulas. The class of Archimedean copulas gives an important example of one-parameterized family of copulas, and its notion as well as its generalized notion will be explained in the next section. The Kendall's tau, on the other hand, is one of frequently utilized measures of concordance between random variables, which is thus expected to be relevant to the concept of copula. Indeed the formula of Kendall's tau in terms of copulas is already known. Our contribution is then to extend this formula to the case of GA copulas. We exhibit some examples of calculation employing this newly obtained formula, and consider its application to the information management. We hope that our research will enhance the applicability of copulas.

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II. Formulas of Copulas

A. Copula

We begin with recalling the definition of copula in the case of bivariate joint distribution.

Definition. A function C defined on $I^2 := [0,1] \times [0,1]$ and valued in I is called a copula if the following conditions are fulfilled.

1) For every $(u,v) \in I^2$,

$$\begin{aligned} C(u,0) &= C(0,v) = 0, \\ C(u,1) &= u \text{ and } C(1,v) = v. \end{aligned} \tag{1}$$

2) For every $(u_i,v_i) \in I^2\,(i=1,2)$ with $u_1 \le u_2$ and $v_1 \le v_2$,

$$C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \ge 0\,. \tag{2}$$

The requirement (2) is referred to as the 2-*increasing condition*. We also note that a copula is continuous by its definition.

The well-know result due to A. Sklar [13], who employed the term "copula" almost for the first time, gives the basic property of copulas. We here recall Sklar's theorem in bivariate case, for completeness of our presentation.

Theorem (Sklar's theorem). Let H be a bivariate joint distribution function with marginal distribution functions F and G; that is,

$$\lim_{x\to\infty} H(x,y) = G(y),$$
$$\lim_{y\to\infty} H(x,y) = F(y),$$

Then there exists a copula, which is uniquely determined on *RanF* $\times$ *RanG, such that*

$$H(x,y) = C(F(x),G(y)). \tag{3}$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (3) is a bivariate joint distribution function with margins F and G.

An important class of copulas is given by the so-called Archimedean copulas. We recall for completeness what are the Archimedean copulas. Let ϕ be a convex function defined on I and valued in $[0,\infty]$ such that ϕ is strictly decreasing and verifies $\phi(1) = 0$. Let denote by $\phi^{[-1]}$ the pseudo-inverse of ϕ; that is, $\mathrm{Dom}\phi^{[-1]} = [0,\infty]$, $\mathrm{Ran}\phi^{[-1]} = I$, and

$$\phi^{[-1]} = \begin{cases} \phi^{[-1]} & (0 \le t \le \phi(0)) \\ 0 & (\phi(0) \le t \le \infty) \end{cases}.$$

It is then possible to prove that the function C defined on I^2 by

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)) \tag{4}$$

satisfies the properties (1)(2) in Definition above, and thus C provides a copula.

Copulas of the form (4) are called Archimedean copulas and the function ϕ is called a generator of the copula. The class of Archimedean copula finds a wide range of applications. We here present several examples of Archimedean copulas:

Clayton copula:

$$\phi(t) = \theta^{-1}(t^{-\theta} - 1) \qquad (\theta \in [-1, \infty) \setminus \{0\}).$$

$$C(u,v) = \left(\max\{u^{-\theta} + v^{-\theta} - 1, 0\}\right)^{-\frac{1}{\theta}}.$$

Gumbel copula:

$$\phi(t) = (\log t)^{\theta} \qquad (\theta \in [1, \infty)).$$

$$C(u,v) = \exp(-[(-\log u)^{\theta} + (-\log v)^{\theta}]^{1/\theta}).$$

Frank copula:

$$\phi(t) = -\log\frac{e^{-\theta t}-1}{e^{-\theta}-1} \qquad (\theta \in (-\infty, \infty) \setminus \{0\}).$$

$$C(u,v) = -\frac{1}{\theta}\log\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right).$$

There are many other Archimedean copulas. For more details, we refer to a book by Nelsen [8].

B. Generalized Archimedean Copulas

In 2007, F. Durante, J. J. Quesada-Molina, and C. Sempi [14] discovered an interesting generalization of Archimedean copulas, which is defined as follows.

Let ϕ be a convex and strictly decreasing function defined on I and valued in $[0, \infty]$, and let ψ be a continuous and decreasing function with $\psi(1) = 0$ defined on I and valued in $[0, \infty]$. Suppose in addition that $(\psi - \phi)$ is increasing on I. For such ϕ, ψ, the function $C_{\phi,\psi}$ defined on ^{2} by

$$C_{\phi,\psi}(u,v) = \phi^{[-1]}(\phi(\min\{u,v\}) + \psi(\max\{u,v\})) \tag{5}$$

provides a copula. It is easy to see that if $\phi \equiv \psi$, then the copula $C_{\phi,\phi}$ is reduced to the original Archimedean copula (4) with the generator ϕ. In this sense, the copula of the form (5) is called a generalized Archimedean copula. Here we briefly recall the proof that (5) indeed gives a copula (see also [14]).

It is immediate to see that $C_{\phi,\psi}$ in (5) verifies (1); it suffices to show that $C_{\phi,\psi}$ fulfills the 2-increasing condition (2).

Let $(u_i, v_i) \in I^2\,(i = 1,2)$ be $u_1 \le u_2$ and $v_1 \le v_2$. Define $R = [u_1, u_2] \times [v_1, v_2]$. Suppose first that $R \subset (u,v) \in I^2 \mid u \ge v\}$. Then we have

$$V_{C_{\phi,\psi}}(R) \coloneqq C_{\phi,\psi}(u_1,v_1) - C_{\phi,\psi}(u_1,v_2) - C_{\phi,\psi}(u_2,v_1) + C_{\phi,\psi}(u_2,v_2)$$

$$= \phi^{[-1]}(s_1) - \phi^{[-1]}(t_1) - \phi^{[-1]}(t_2) + \phi^{[-1]}(s_2),$$

where we have put

$$s_1 = \phi(v_1) + \psi(u_1), \qquad s_2 = \phi(v_2) + \psi(u_2),$$

$$t_1 = \phi(v_2) + \psi(u_1), \qquad t_2 = \phi(v_1) + \psi(u_2).$$

Since $\min\{t_1, t_2\} \ge \min\{s_1, s_2\}$, $t_1 + t_2 = s_1 + s_2$, and $\phi^{[-1]}$ is a convex function, we infer that $V_{C_{\phi,\psi}}(R) \ge 0$ and $C_{\phi,\psi}$ verifies the 2-increasing condition in this case. Thanks to the symmetry of $C_{\phi,\psi}$, we also find that $C_{\phi,\psi}$ satisfies the 2-increasing condition in the case of $R \subset \{(u,v) \in I^2 \mid u \le v\}$.

Next, consider the case that the diagonal of R lies on the diagonal of the unit square; namely, $u_1 = v_1$ and $u_2 = v_2$. If $u_1 = 0$, then $V_{C_{\phi,\psi}}(R) = \phi^{[-1]}(\phi(u_2) + \psi(u_2)) \ge 0$ and we are done. Assume that $u_1 > 0$. It follows that

$$V_{C_{\phi,\psi}}(R) = \phi^{[-1]}(s_1) - \phi^{[-1]}(t_1) - \phi^{[-1]}(t_2) + \phi^{[-1]}(s_2),$$

where we have put this time

$$s_1 = \phi(u_1) + \psi(u_1), \qquad s_2 = \phi(u_2) + \psi(u_2),$$

$$t_1 = \phi(u_1) + \psi(u_2) = t_2.$$

We learn that $\min\{t_1, t_2\} \ge \min\{s_1, s_2\}$ and $t_1 + t_2 \ge s_1 + s_2$, in view of the condition that $(\psi - \phi)$ is increasing on I. Since $\phi^{[-1]}$ is a convex function, we infer that $C_{\phi,\psi}$ verifies the 2-increasing condition also in this case.

Finally, since every rectangle R in I^2 can be decomposed into the union of rectangles of above three types and $V_{C_{\phi,\psi}}(R)$ can be computed as a sum of these decompositions, we conclude that $C_{\phi,\psi}$ satisfies the 2-increasing condition.

This proves that the function $C_{\phi,\psi}$ defined on I^2 by (5) indeed gives a copula.

We mention some nontrivial examples of a generalized Archimedean copula, which are presented in [14].

Example 1. Let $\phi(t) = -\log t$, $\psi(t) = -\log t^{\alpha}$ with $\alpha \in [0,1]$. Then we find that the corresponding copula $C_{\phi,\psi}$ is

$$C_{\phi,\psi} = \begin{cases} uv^{\alpha} & (u \le v) \\ u^{\alpha}v & (u \ge v) \end{cases}.$$

This family is a member of Cuadras-Augé family of copulas.

Example 2. Let $\phi(t) = \alpha(1-t)\ (\alpha \ge 1)$ and let $\psi(t) = 1 - t$.

The corresponding copula $C_{\phi,\psi}$ is

$$C_{\phi,\psi}(u,v) = \max\left\{0, \min\{u,v\} - \frac{1}{\alpha}(1 - \max\{u,v\})\right\}$$

$$= \begin{cases} \frac{1}{\alpha}(\alpha \min\{u,v\} + \max\{u,v\} - 1) \\ \quad (\text{if } \alpha \min\{u,v\} + \max\{u,v\} \ge 1) \\ 0 \qquad (\text{otherwise}). \end{cases}$$

Example 3. Let $\phi(t) = 1 - t$ and for every $0 \le \alpha \le 1$,

$$\text{let } \psi(t) = \begin{cases} \frac{\alpha}{2} & (0 \le t \le \frac{\alpha}{2}) \\ \alpha - t & (\frac{\alpha}{2} \le t \le \alpha). \\ 0 & (\alpha \le t \le 1). \end{cases}$$

The corresponding copula $C_{\phi,\psi}$ is

$$C_{\phi,\psi}(u,v) = \begin{cases} \max\{0, u + v - \alpha\} & (0 \le u,v \le \alpha) \\ \min\{u,v\} & (\text{otherwise}). \end{cases}$$

This is a member of the Mayor-Torrens family of copulas.

C. Kendall's Tau

To quantitatively estimate various dependent relations, several measures of associations have been introduced so far. As widely known examples, we recall the population version of Kendall's tau and Spearman's rho. We here restrict ourselves to the investigation of the population version of Kendall's tau, which is denoted by τ.

It is known that τ can be represented in terms of copulas. Precisely stated, let X and Y be continuous random variables whose copula is C. Then we have

$$\tau_{X,Y} = \tau_C = 4\iint_{I^2} C(u,v)dC(u,v) - 1$$
$$= 1 - 4\iint_{I^2} \frac{\partial C}{\partial u}(u,v)\frac{\partial C}{\partial v}(u,v)dudv.$$

For the proof, we refer to the book of Nelsen [8].

If the copula C is Archimedean of the form (4), then τ_C is further computed as follows.

$$\tau_C = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt \tag{6}$$

We quickly recall the proof of formula (6) for completeness of our presentation. Our reasoning is so organized that the generalization to GA copulas will be easy.

To see (6), it suffices to consider the case that ϕ is differentiable and thus we compute

$$\iint_{I^2} \frac{\partial C}{\partial u}\frac{\partial C}{\partial v}dudv$$
$$= 2\iint_{\{u\le v\}} \frac{\phi'(u)\phi'(v)}{\left(\phi'\left(\phi^{[-1]}(\phi(u)+\phi(v))\right)\right)^2}dudv.$$

Performing the change of variables $(u,v) \to (u,t)$, where $\phi(t) = \phi(u) + \phi(v)$ on $\{u \le v\}$, we infer that

$$\iint_{\{u\le v\}} \frac{\phi'(u)\phi'(v)}{\left(\phi'\left(\phi^{[-1]}(\phi(u)+\phi(v))\right)\right)^2}dudv$$
$$= \iint_{\{\phi^{[-1]}(\phi(u)+\phi(v))\le t\le u\}} \frac{\phi'(u)}{\phi'(t)}dtdu$$
$$= \int_0^1 \frac{dt}{\phi'(t)}\left(\int_t^1 \phi'(u) - \int_{t^*}^1 \phi'(u)\right)du$$
$$= \int_0^1 \frac{\phi(t)-\phi(t^*)}{\phi'(t)}dt = \frac{1}{2}\int_0^1 \frac{\phi(t)}{\phi'(t)}dt,$$

where t^* is determined by $\phi(t^*) = 2^{-1}\phi(t)$.

Now, the new result in this article is that we extend the formula (6) to the case of GA copulas. Our main establishment is now read as follows. See also [15].

Theorem. Let X and Y be continuous random variables whose copula $C_{\phi,\psi}$ is a GA copula given by (5). Then the Kendall's tau of X and Y is expressed as

$$\tau_C = 1 + 8\int_0^1 \frac{\phi(t)}{\phi'(t^*)}dt, \tag{7}$$

where t^* *is defined through* $\phi(t^*) + \psi(t^*) = \phi(t)$.

We remark that if $\phi \equiv \psi$, then $\phi(t^*) = 2^{-1}\phi(t)$ and the formula (6) is recovered from (7).

Proof. In view of the approximating argument, it suffices to prove the case that both ϕ, ψ are differentiable. We compute

$$\iint_{I^2} \frac{\partial C_{\phi,\psi}}{\partial u}\frac{\partial C_{\phi,\psi}}{\partial v}dudv$$
$$= \iint_{\{u\le v\}} \frac{\phi'(u)\psi'(v)}{\left(\phi'\left(\phi^{[-1]}(\phi(u)+\psi(v))\right)\right)^2}dudv$$
$$+ \iint_{\{u\ge v\}} \frac{\phi'(v)\psi'(u)}{\left(\phi'\left(\phi^{[-1]}(\phi(v)+\psi(u))\right)\right)^2}dudv$$
$$= 2\iint_{\{u\le v\}} \frac{\phi'(u)\psi'(v)}{\left(\phi'\left(\phi^{[-1]}(\phi(u)+\psi(v))\right)\right)^2}dudv.$$

Performing the change of variables $(u,v) \to (u,t)$, where $\phi(t) = \phi(u) + \psi(v)$ on $\{u \le v\}$, we infer that

$$\iint_{\{u\le v\}} \frac{\phi'(u)\psi'(v)}{\left(\phi'\left(\phi^{[-1]}(\phi(u)+\psi(v))\right)\right)^2}dudv$$
$$= \iint_{\{\phi^{[-1]}(\phi(u)+\psi(v))\le t\le u\}} \frac{\phi'(u)}{\phi'(t)}dtdu$$
$$= \int_0^1 \frac{dt}{\phi'(t)}\left(\int_t^1 \phi'(u) - \int_{t^*}^1 \phi'(u)\right)du$$
$$= \int_0^1 \frac{\phi(t)-\phi(t^*)}{\phi'(t)}dt,$$

where t^* is determined by $\phi(t^*) + \psi(t^*) = \phi(t)$.

This finishes the proof of Theorem.

We show how this formula (7) works through the before mentioned examples.

Example 1 (continued). Since $\phi(t) = -\log t$, $\psi(t) = -\log t^\alpha$, it is immediate that $t^* = t^{1/(\alpha+1)}$. Consequently we have

$$\tau = 1 + 8\int_0^1 t\log t^{1-1/(\alpha+1)}dt = 1 - \frac{2\alpha}{\alpha+1}.$$

Example 2 (continued). Since $\phi(t) = \alpha(1-t)$, $\psi(t) = 1-t$, we find that $1 - t^* = \alpha(1-t)/(\alpha+1)$. We thus obtain

$$\tau = 1 + 8\int_0^1 \frac{-(1-t)}{\alpha+1}dt = 1 - \frac{4}{\alpha+1}.$$

Example 3 (continued). Calculating similarly, we deduce that

$$t^* = \begin{cases} \frac{1}{2}(t+\alpha) & (0 \le t \le \alpha) \\ t & (t \ge \alpha). \end{cases}$$

We obtain

$$\tau = 1 + 8\int_0^1 -(1-t)\,dt = -3.$$

We note that the same results are also obtained by direct calculation. However, the computation becomes easier through the use of our formula.

III. Conclusion

Copulas are widely employed in every area where certain nonlinear relations are concerned. In this research, we have derived the formula of the Kendall's tau for the family of generalized Archimedean copulas. The Kendall's tau is a well known measure of concordance between random variables. The formula brings to us an alternative way of computation of the quantity, which is important in measuring the concordance relation. Examples show that the computation through this formula is rather handy. We hope that our achievement will enrich the application of copulas, as well as its application to the information management which needs entail the nonlinear dependence relation between several factors.

On the other hand, the theory of copulas is still far from complete. One sees that the time variable does not appear in the definition of copulas, although the dependence relation between random variables usually changes along with the time. Indeed, one of the authors recently introduce the notion of evolution of copulas [16], [17], which proclaims that the dependence relation evolves according to the time variable. It is to be noted that the evolution of copula is different from the theory of dynamic copula [18]. It is interesting to know how a family of (generalized) Archimedean copulas is related to this new concept of evolution of copulas. Our next project is now advancing in this issue.

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