State Estimation and Error Analysis of a Single State Dynamic System with Sensor Data Using Kalman Filter

Toshak Singhal, Akshat Harit, and D. N. Vishwakarma

***Abstract*—Kalman Filter is used in system estimation applications today like state estimation, load flow analysis, harmonic estimation, digital signal processing, sensor integration, Navigational Systems, etc. In using a Kalman Filter the user has to give the parameters relating the estimates of process and measurement noise along with system state modeling. The values of process and measurement noise covariance are usually not available beforehand and have to be estimated, usually by hit or trial method. This involves heavy computation, as two variables have to be estimated for optimal filtering independently. For multi-state systems this value further increases the computation time. This paper presents the application of Kalman Filter to a simple one state problem. This paper, through using simulations, finds relationships between the two different parameters Q (Process Noise Covariance) and R (Measurement Noise Covariance). This results in reduction of computation time. The proposed scheme's low complexity and robustness makes it practical for real implementations.**

***Index Terms*—Estimator, kalman error analysis, kalman filter, kalman optimization, measurement noise covariance, state estimation, process noise covariance.**

I. Introduction

Kalman Filter is a digital filter used to filter noise on a series of measurements observed over a time interval. Kalman Filter named after Rudolf E. Kálmán who published a paper "A new approach to linear filtering and prediction problems"[1] in 1960, recent advancements have been made and various other filters such as Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) have been derived from it. It is an algorithm used to solve the linear quadratic Gaussian (LQG) estimation problem. It operates recursively on the data stream of a dynamic system to give an optimum estimate of the current system state. It has numerous applications in various fields like Power System state estimation [2], [3], Aircraft Guidance and navigational control systems. The Kalman filter algorithm is based on two steps; first the prediction step in which the current estimate of state variables, with random noise included is given. The prediction step only involves the data measurement before the time at which system state is to be calculated. These estimates are used along with the measurement, with random Gaussian noise, to give the correct state of the system. The algorithm works by using a weighted average model on the predicted value and the current value. The more certain measurement is given more weight. The filter works in the discrete time domain. Implementations are available for continuous time version, called Kalman-Bucy filter. Extended Kalman Filter (EKF) can be used for Extended Kalman Filter. Another variant, the Unscented Kalman Filter (UKF) [4] is used when state transition and observation models are highly non-linear i.e. cases in which EKF gives poor performance. Also Kalman filter has been proved to give excellent results in the sensor data fusion [5] and there have been various algorithms for it using Kalman Filter and Fuzzy logic specially. Kalman filter in sensor data fusion treats one sensor data as measurement and other as prediction. It has been very frequently used to integrate GPS (Global Positioning System) and IMU (Inertial Measurement Unit; Gyroscope, Accelerometer, Magnetometer) [6] systems employed in various vehicles both Airborne and terrestrial automated vehicles. Kalman filter has also been successfully implemented in the coal flow problem [7] in Thermal Power plant to optimize the coal input for given power production. However the modeling of the problem is difficult in practical implementation especially but Kalman filter is known to give very satisfactory and reliable results in various practical problems of Electrical and Electronics engineering.

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II. Nomenclature

- x_k System State Matrix
- wk Process Noise
- zk Measurement Result Matrix
- vk Measurement Noise
- Φk State Transition Matrix
- Pk State Error Covariance Matrix
- Hk Measurement transition Matrix
- Kk Kalman Gain
- Q Process Noise Covariance
- R Measurement Noise Covariance
- E Expectation Operator

III. System Model

System modeled in this paper is a single state system. For the sake of simplicity we have only focused on Kalman Filter's performance and computation, so we have assumed state transition matrix and measurement state matrix to be unity. The process noise added is White Gaussian noise with signal to noise ratio equal to -15. Similarly, the measurement noise is also White Gaussian noise with signal to noise ratio equal to -15. Now system equation equations can be given as

$$X_{k+1} = x_k + W_k \tag{1}$$

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$$Z_k = x_k + V_k \tag{2}$$

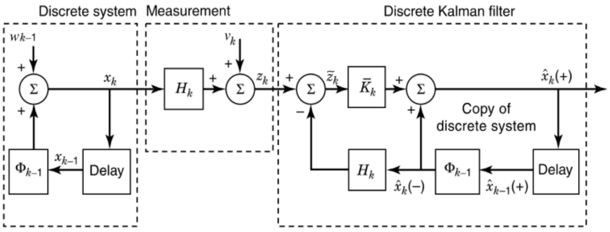


Fig. 1. Block diagram

IV. Kalman Filter Mathematical Formulation

A. Equations [8]

1) System dynamic model

$$z_k = H_k x_k + v_k \tag{3}$$

$$v_k = n(0, Q_k) \tag{4}$$

The above equations represent how our system is modeled. Φ is the state transition matrix. w_k is the process noise. It is assumed to be zero mean Gaussian noise

2) Measurement model

$$z_k = H_k x_k + v_k \tag{5}$$

$$v_k = n(0, Q_k) \tag{6}$$

It is assumed that measurement is related to state by the above equation, where H is the measurement sensitivity matrix and v_k is the measurement noise. This is also assumed to be white Gaussian noise (zero mean).

3) Initial conditions

$$E(x_0) = \hat{x}_k \tag{7}$$

$$E\left(\hat{x}_0 \hat{x}_0^T\right) = P_0 \tag{8}$$

P_0 is the priori covariance matrix. It is initialized as above. Expectation of x is assumed to be optimal estimate of initial value.

4) Independence assumptions

$$E\left(w_k v_j^T\right) = 0 \ for\, k\, and\ j \tag{9}$$

The process noise and measurement noise are assumed to be independent of each other

5) State estimate extrapolation

$$\hat{x}_k = \Phi_k \hat{x}_{k-1}(+) \tag{10}$$

This equation represents the prediction step of Kalman Filter. As we do not know about the particular value of noise signal and any other estimate of the system, we take the prediction value using our state transition matrix. The above can include control input as well, if necessary or required.

6) Error covariance extrapolation

$$p_k(-) = \Phi_{k-1} p_{k-1}(+) \Phi_{k-1}^T + Q_{k-1} \tag{11}$$

This step models the effect of time on covariance matrix of estimation certainty as a function of previous posteriori value $P_{k-1.}$

7) State Estimate observational update

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k\left[Z_k - H_k \hat{x}_k(-)\right] \tag{12}$$

This equation gives the Kalman output of the current signal. K_k is the Kalman gain, which represents the relative weight of the past measurements, based on system modeling and on the current measure input through the sensor measurement

8) Error covariance update

$$P_k(+) = \left[1 - \bar{K}_k H_k\right] P_K(-) \tag{13}$$

Error covariance is updated in this equation using Kalman gain and posteriori value. This implements the effect that conditioning on the measurement has on the covariance matrix of estimation uncertainty.

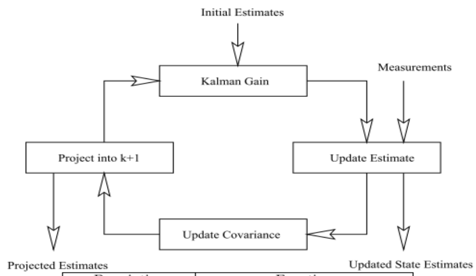
9) Kalman gain matrix

$$\bar{K}_k = P_K(-) H_k^T \left[H_k P_k(-) H_k^T + R_k\right]^{-1} \tag{14}$$

In this equation the Kalman Gain (K_k) is updated using the new values generated during this particular set of measurements.

As all the parameters are recursively computed based on its previous value, the previous on its previous value till initial condition, the Kalman filter incorporates the information obtained by all the previous values in its prediction. It does so without actually storing that data and uses simple equations in a loop, making it computationally inexpensive. It is also necessary to point out that Kalman gain and error covariance equations are independent of actual observations. These parameters can be used to obtain preliminary information about the estimator performance. Since algorithm is recursive so it can be implemented computationally as the length of the matrix doesn't increase with time and this is very helpful in the problems with multi-state or multidimensional problems.

B. Algorithm for the Kalman Implementation in This Paper



Description	Equation
Kalman Gain	$K_k = P'_k H^T \left(H P'_k H^T + R\right)^{-1}$
Update Estimate	$\hat{x}_k = \hat{x}'_k + K_k (z_k - H\hat{x}'_k)$
Update Covariance	$P_k = (I - K_k H) P'_k$
Project into $k+1$	$\hat{x}'_{k+1} = \Phi \hat{x}_k$ $P_{k+1} = \Phi P_k \Phi^T + Q$

Fig. 2. Flowchart

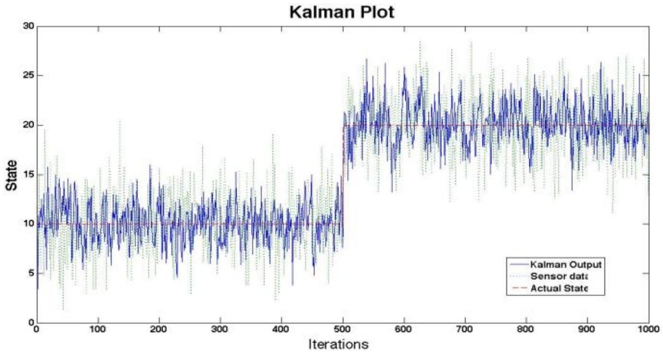


Fig. 3. R/Q=1

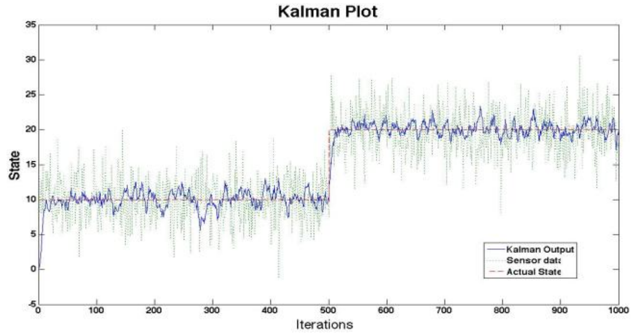


Fig. 4. R/Q=20

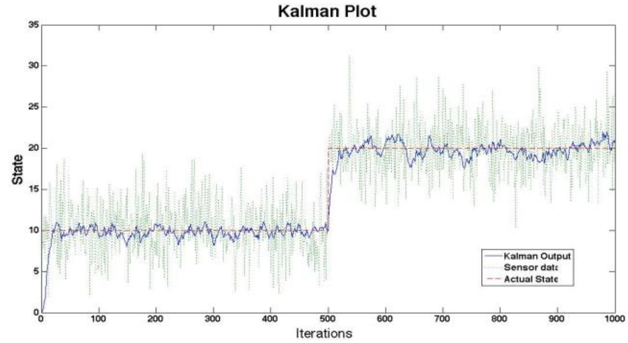


Fig. 5. R/Q=80

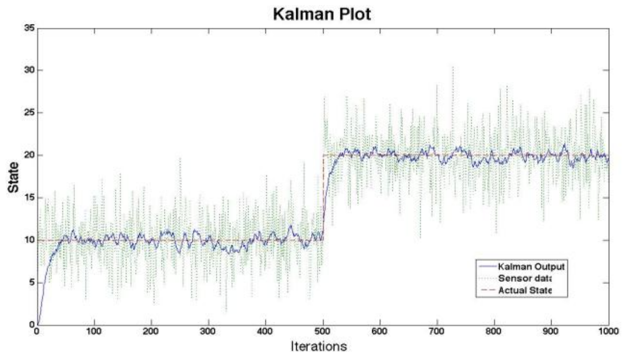


Fig. 6. R/Q=120

V. Computer Simulation Tests

A modeling of a simple 1-dimensinal dynamic system with both H and Φ=1 was done in MATLAB; the Kalman Filter was run for 1000 iterations in each simulation. The signal supplied was a discrete step wave of amplitude 10 and 20 units in the first and second parts of the iterations. Noise was added to both measurement and process with signal to noise ratio= -15db. The simulations were run for different values of Q and R. Tests were conducted on the MATLAB software.

VI. Simulation Results

In Fig. 3 to Fig. 6, Q is kept constant and value of R is changed for each simulation. In Fig. 3, with Q=0.01 and R=0.01, we find that the Kalman output (blue line) of the measurement signal (green line), the sensor data converges quickly to the signal (red line) i.e. it reaches the true state in less number of iterations. However the Kalman output has very high ripple content and output data may be treated as inconclusive. Although better than the noisy input, the Kalman filter output does not give adequate results. In the next Fig. 4 with Q=0.01 and R=20*0.01, it was seen that Kalman output shows a much better noise free data, however this time it take more number of iterations to converge. Further increasing the ratio R/Q to 80, we have good Kalman filter output with noise reduced between amplitudes ±2 from the previous ±3. It is observed that further increase in the ratio R/Q from 80 the ripple or noise decrement is not significant however on the other hand time for convergence i.e. number of iteration taken by Kalman to estimate the system within the expected confidence intervals increase rapidly. In the Fig. 4 we can see that the noise filtered is not of significant decrement but the time for convergence of the estimator has changed from 20 to 40 iterations. Now further increase will offset by late convergence of Kalman filter without significant noise filtering so an optimal value of the ratio R/Q was obtained from plotting the error versus R/Q depending upon the confidence intervals required. In Fig. 7, we find that using error function that is sum of square of all vector elements subtracted from the actual state. Error function can be changed depending on whether we require a filter that has fast convergence rate or lesser noise ratio. Depending on the requirements we can modify our error function accordingly by assigning different weights to error before and after convergence.

VII. Estimation of Process Noise (Q) and Measurement Noise(R) Covariance

In the Kalman filter, the weight of the current data and recursively computed predicted value is calculated on the bases of two matrices supplied by the user. These matrices, represented in equation (4) and (6) as Q and R, are process noise and measurement noise covariance respectively. These matrices depend on noise in signal. However as we do not know the noise distribution exactly prior to the experiment, the estimation of Q and R can be done using various algorithms like Auto-covariance Least Square Method [9], [10] and Riccati's Equation [8]. However, frequently hit and trial method is used.

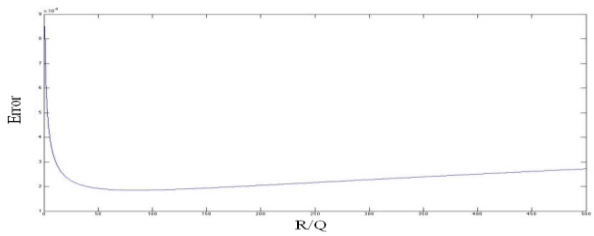


Fig. 7. Error versus R/Q

TABLE I

	Q	R/Q	Iterations	Noise Amp (±)
Fig 3	0.01	1	4	7
Fig 4	0.01	20	10	5
Fig 5	0.01	80	35	3
Fig 6	0.01	120	50	1.5

VIII. Error Analysis and Optimizing Algorithm

Since in the Kalman Filter main problem lies in Optimizing and calculating the error with respect to the actual state, in this paper we have applied an Algorithm that considers both the convergence period i.e. the number of iterations and over the time deviation from the actual or true state. The error function used in this paper is:

$$\text{Error}(R/Q) = \sum(\text{Estimated-Actual})^2 \quad (15)$$

Where the Error plotted is the function of R/Q. As seen from the Fig. 7 we conclude that the optimal value is at R/Q approximately equal to 80. This Algorithm since takes into account of all the data of the state it in fact penalises the error function for both the optimizing fields i.e. the in late convergence and lesser noise reduction. It is found that while using this function we were able to analyse and hence optimize our Kalman filter for single state.

IX. Discussion

Based on the results of simulations obtained, the Kalman filter (and its derivatives EKF, UKF) can be used in many different applications such as Artificial Intelligence, Digital Signal Processing, Image Processing, Communication Systems, Navigation Systems, Sensor Integration etc. Many of these systems are one state dynamic system so the algorithm used here can be directly applied to the measurements and sensor data by modelling the system correctly. Kalman Filter can be suitably modified and tuned (i.e. R/Q estimation) for any application to give the correct balance between convergence time and noise reduction as stated in this paper. Using the Algorithm in this paper for the Kalman Optimization we have concluded that optimality can be obtained for even multi-state and multi-sensor problems. This allows the Kalman filter to be a ubiquitous tool due to its computational efficiency, low memory requirements and ease of use.

X. Results and Conclusions

The Kalman filter output showed different values of convergence time and noise reduction for varying R/Q. The increasing values of this ratio gave better noise reduction. However this advantage is offset by the increase in convergence time. Hence in any application, based on requirements the correct value of R/Q has to be estimated for the filter to work in an efficient and optimal manner. Based on the results of simulation, the estimation of R/Q using hit and trial method should be very much simplified as we have reduced the two variable (Q and R) estimation problem to one variable (R/Q) estimation problem We have found in our problem that ratio (R/Q) be equal to 80.

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